Making choices in multi-dimensional parameter spaces

PhD thesis defence

Steven Bergner

gruvi graphics + usability + visualization

Model adjustment at different levels

- User-driven experimentation: Use cases for *paraglide*
- Criteria optimization: Lighting design
- Theoretical analysis: Sampling in volume rendering
- Discretizing a region: Lattices with rotational dilation
- Summary and conclusion

Data acquisition and visualization

Turning code into data

- Computer simulation code
- Function abstraction
    - Variables: input, output, and algorithm specific
    - Deterministic code

Biological aggregations

(c) Sareh Nabi Abdolyousefi

Biological aggregations

Input	Output

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Biological aggregations

Input	Output
1D+time model *14 parameters* •attraction, repulsion, and alignment coefficients •turning rates *internal:* •space-time resolution influences cost	

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Biological aggregations

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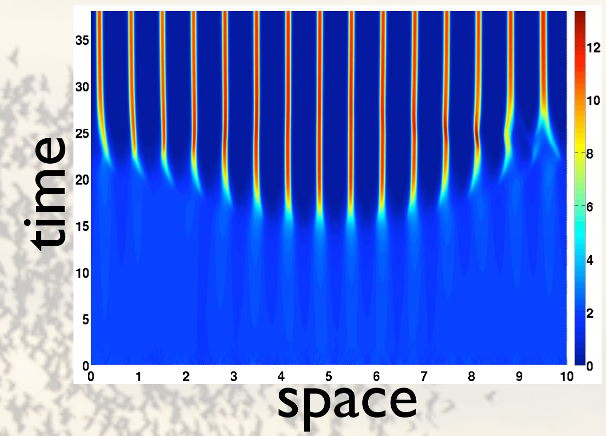
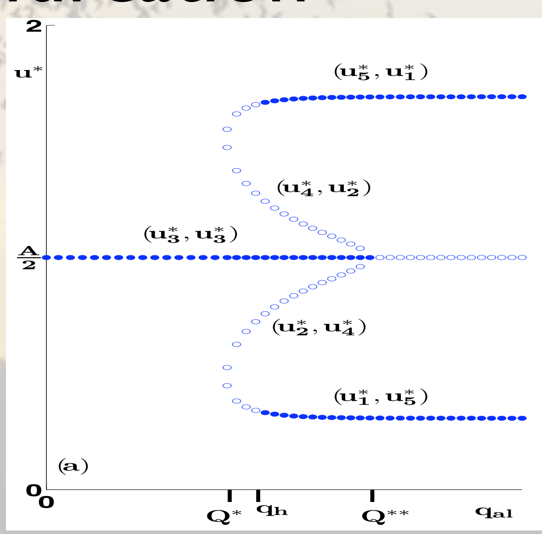
1D+time model *14 parameters* • attraction, repulsion, and alignment coefficients • turning rates *internal:* • space-time resolution influences cost	patterns:  steady state bifurcation and stability: 

More cases

- Parameter space segmentation
- Bio-medical imaging algorithm
- Fuel cell design
- Scene lighting configuration
- Raycasting step size parameter

Paraglide design

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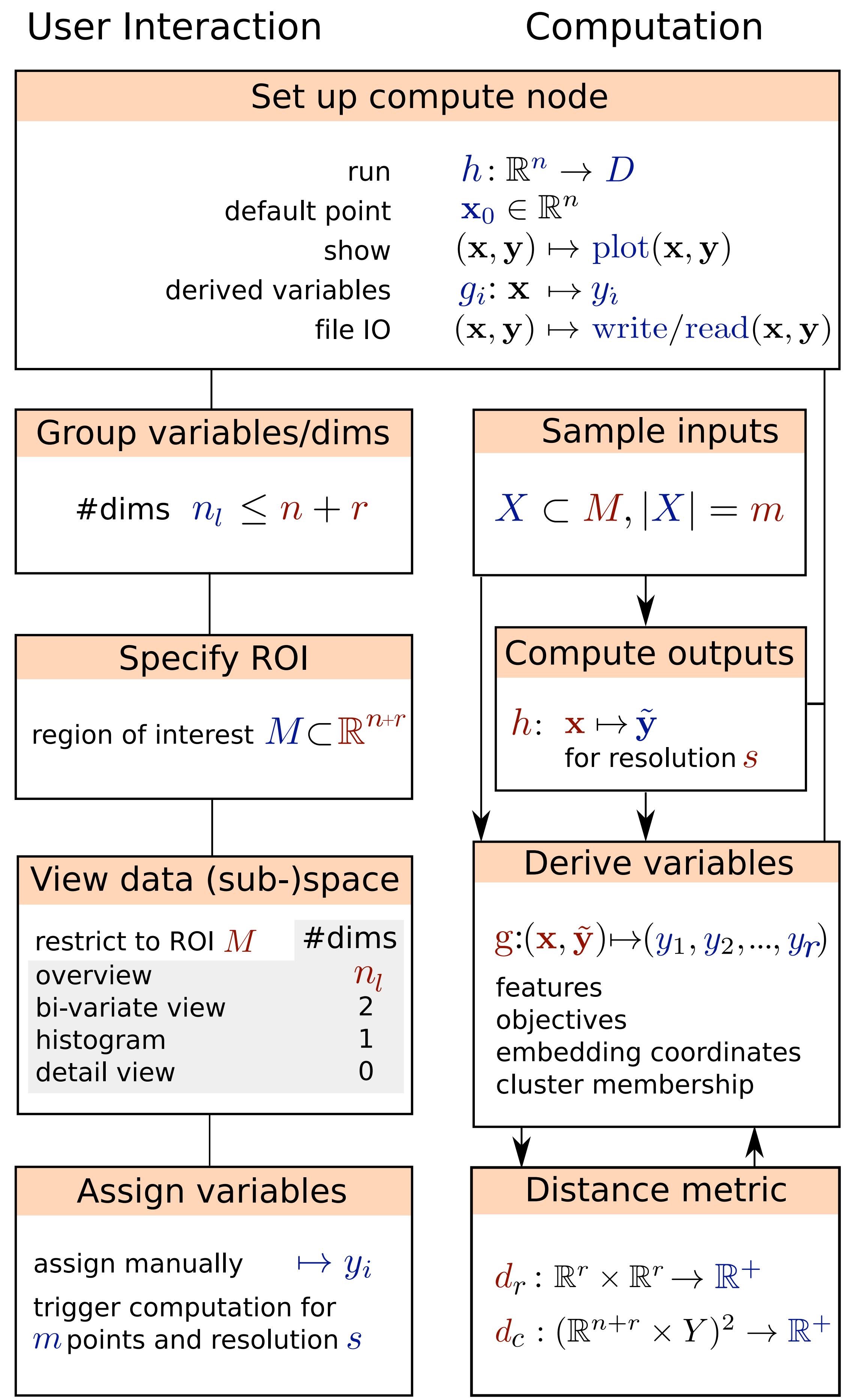

Paraglide design

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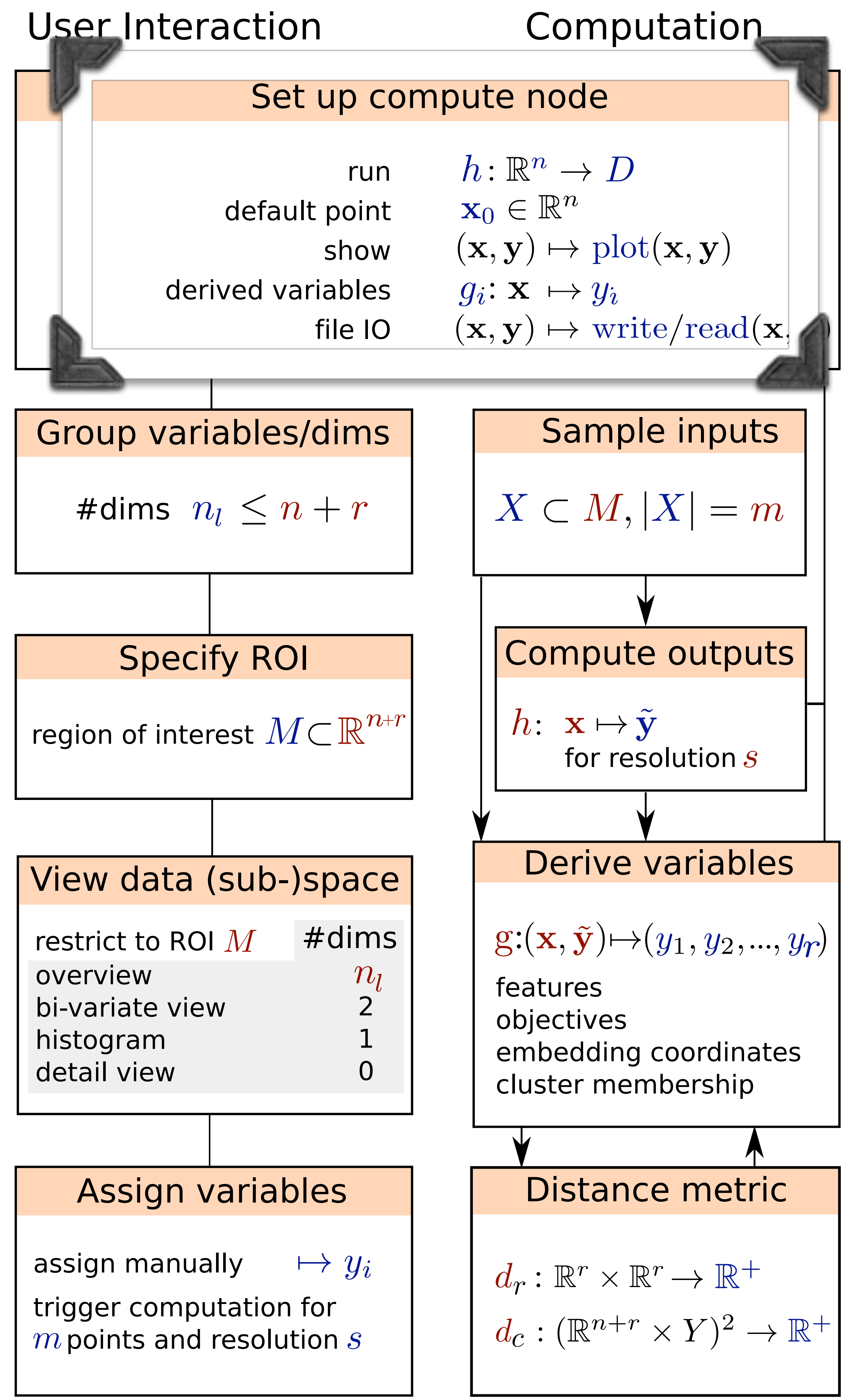

Paraglide design

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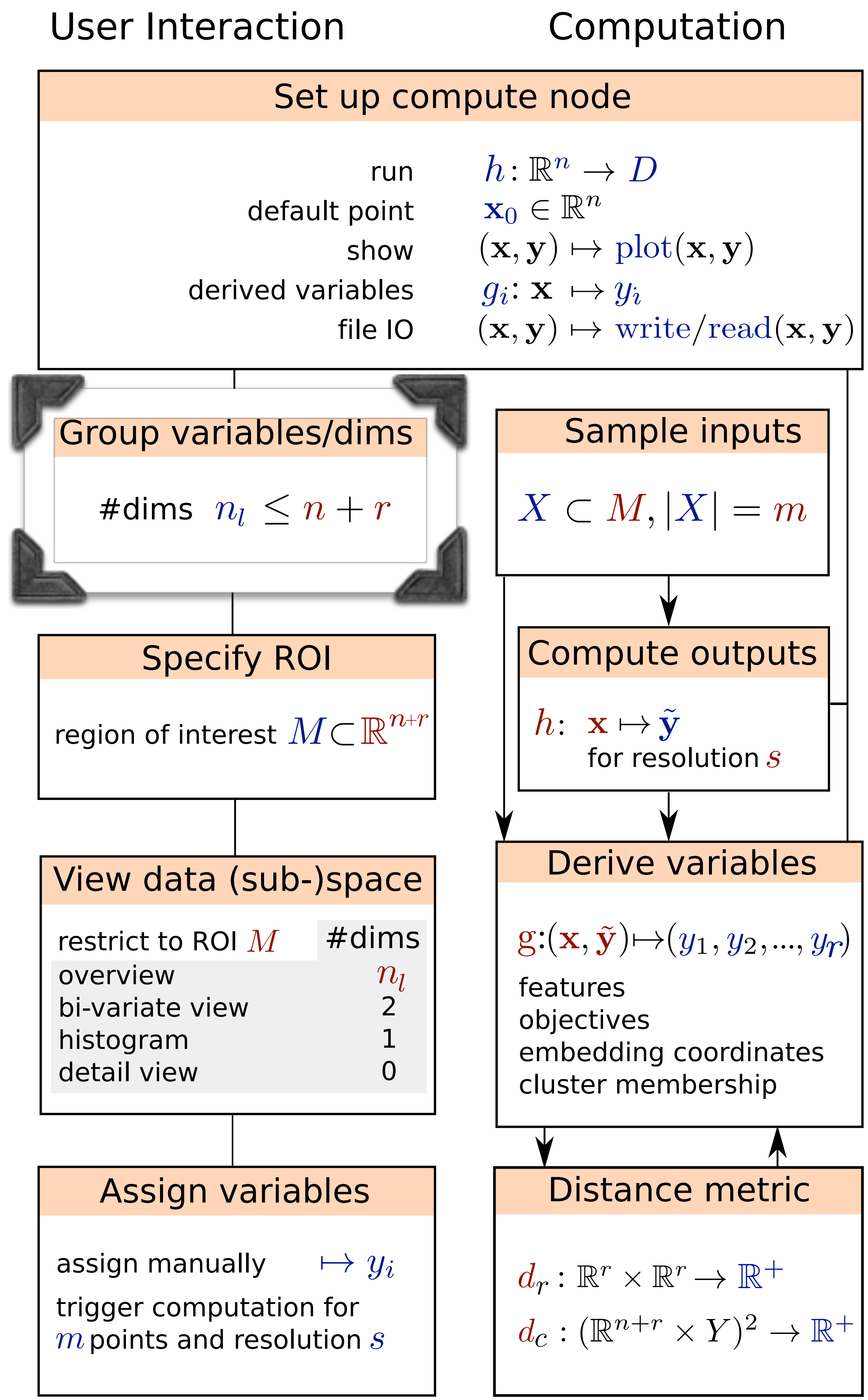

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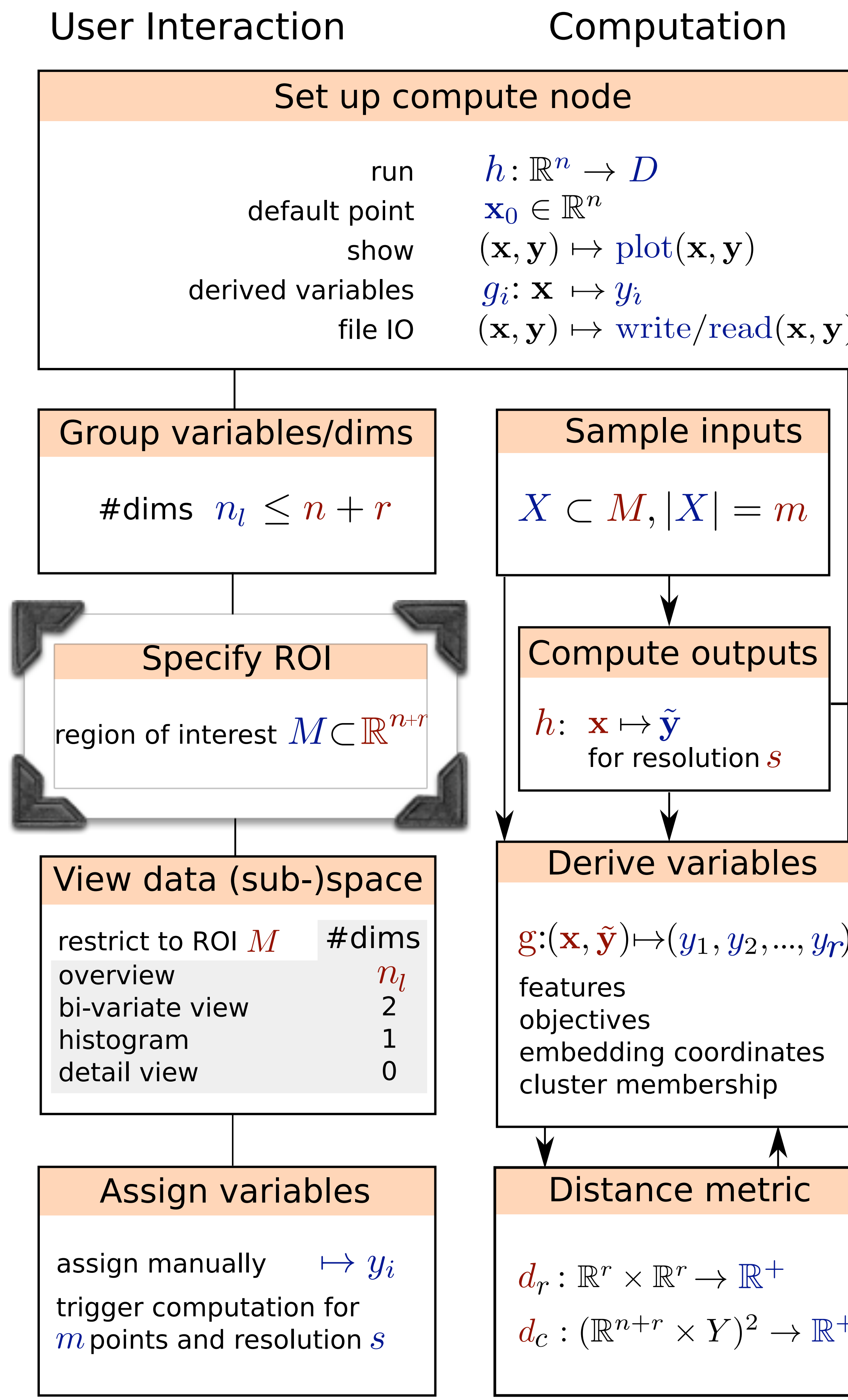

Paraglide design

- Setup compute node
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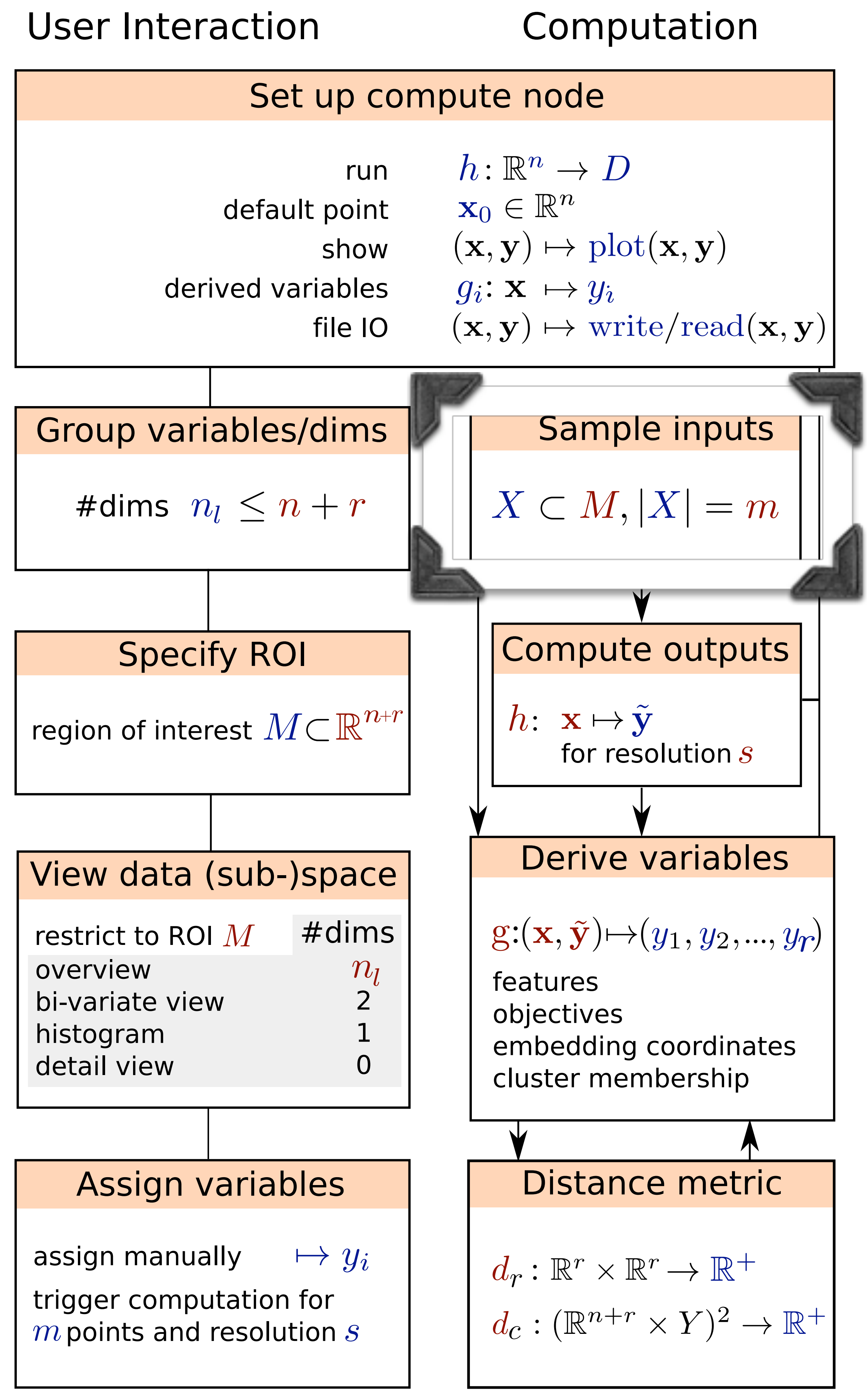
- Choose region
- Sample and compute



Paraglide design

- Setup compute node
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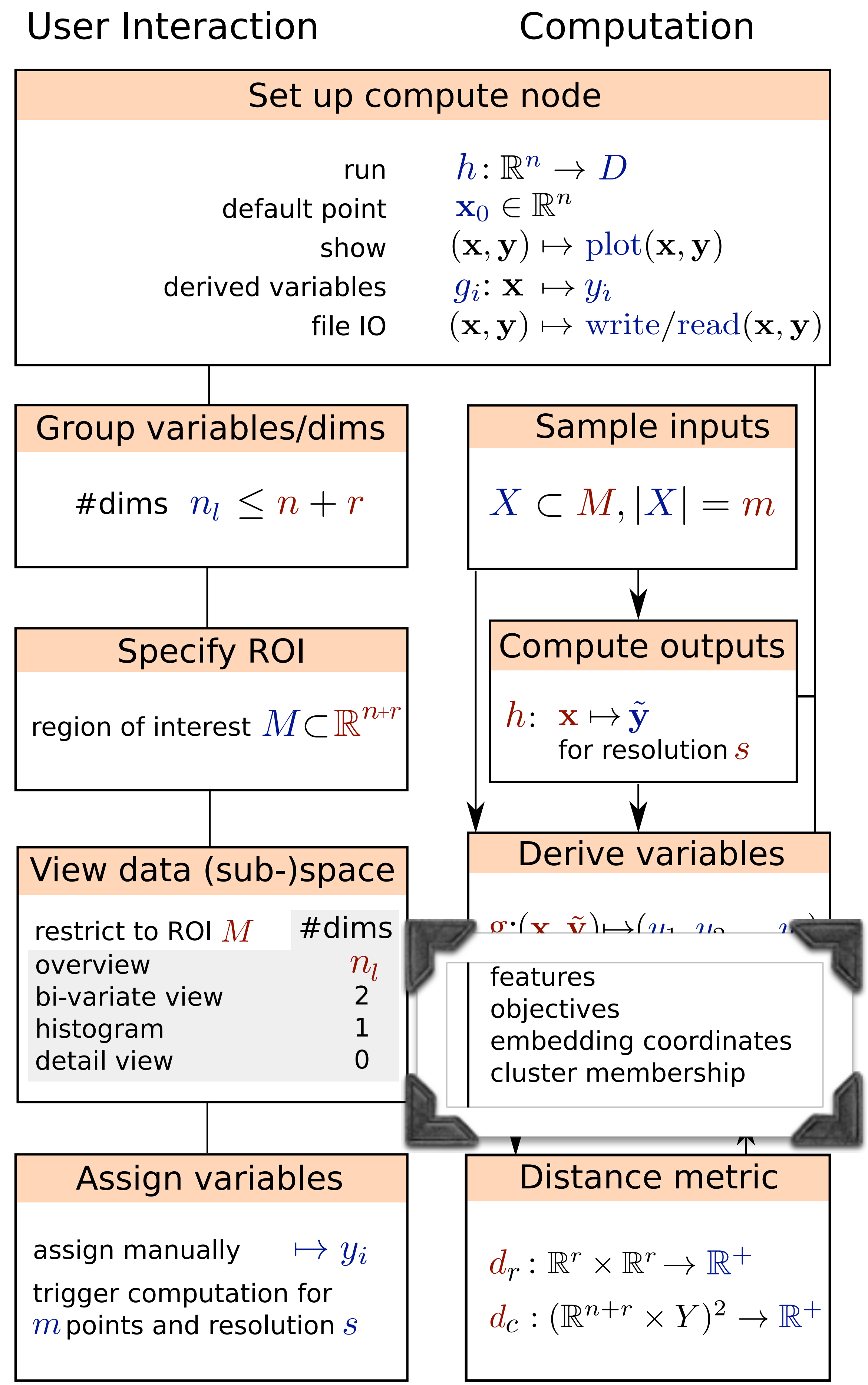
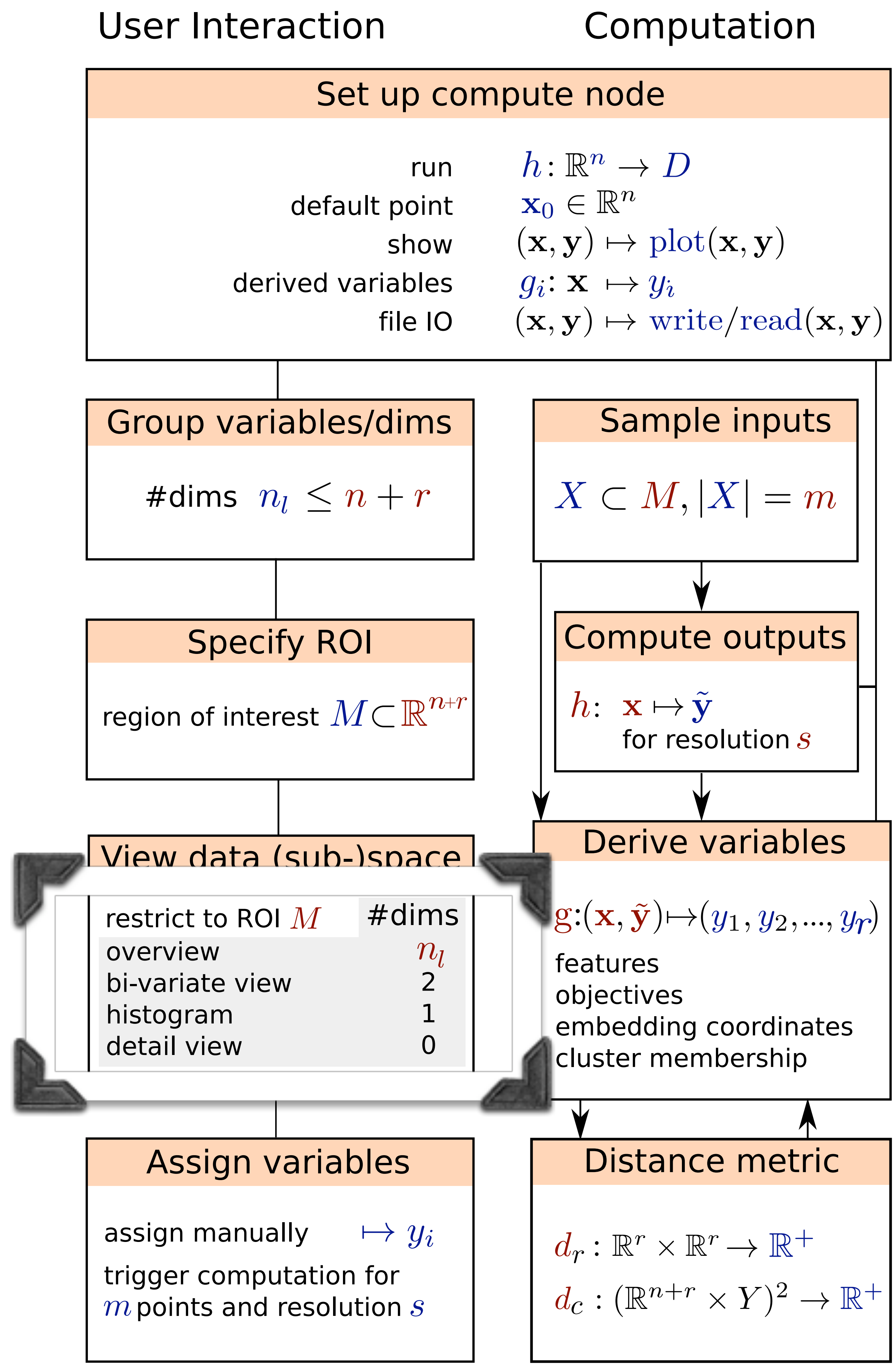
- Compute features



Paraglide design

- Setup compute node
- Choose variables
- Choose region
- Sample and compute
- Compute features
- View, predict, diagnose



Sampling the region of interest

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  - Nested *for*-loops
  - Cost is exponential in #dims

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Paraglide summary

- Longitudinal study showed use of parameter space partitioning
- Requirements informed follow-up projects
- Alternative user interaction
  - Dimensionally reduced slider embedding
  - Mixing board
- Video demo

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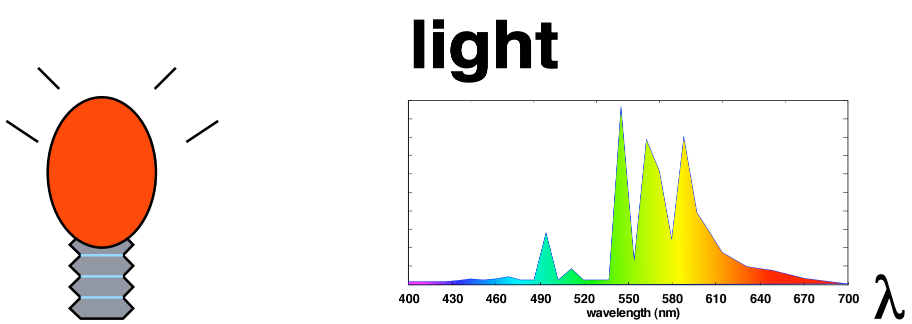
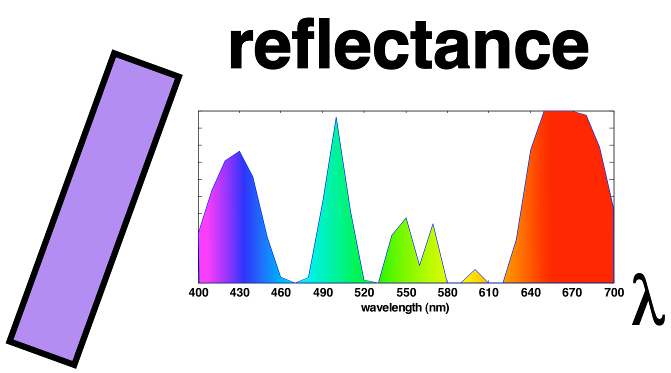
Roadmap

- From light to colour
- Efficient light model
- Designing spectra for lights and materials
- Evaluation
- Applications

From Light to Colour





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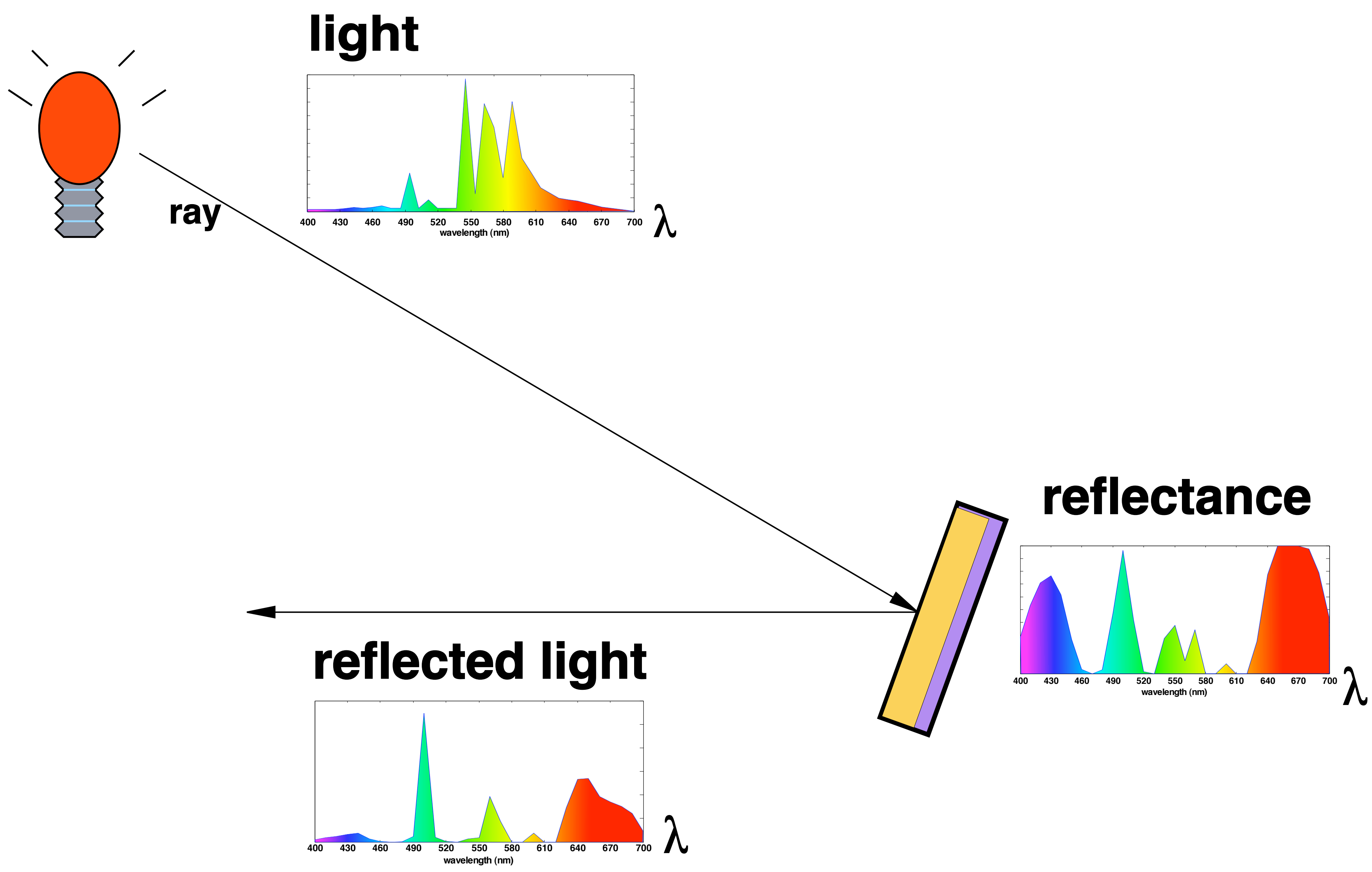

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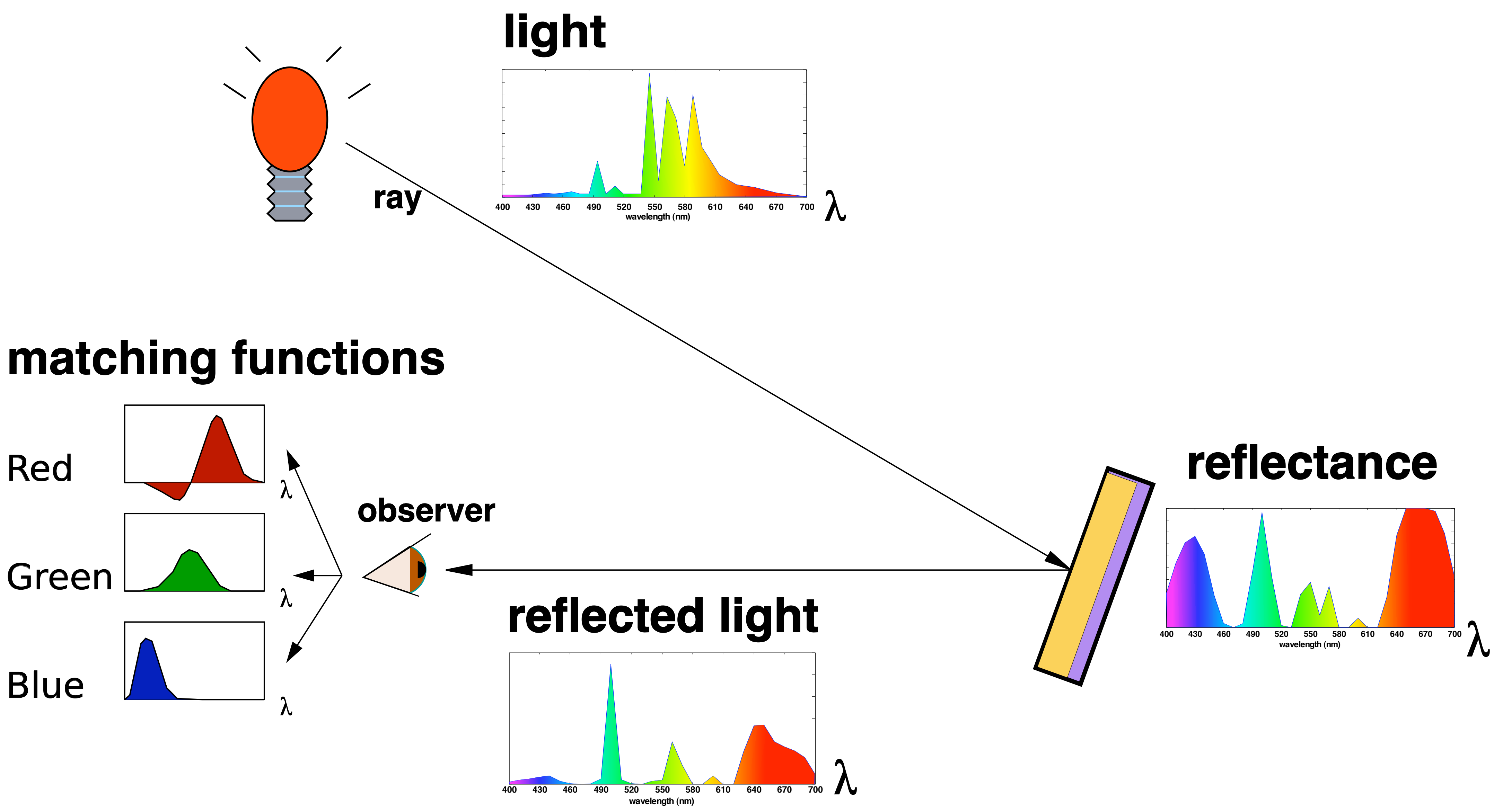

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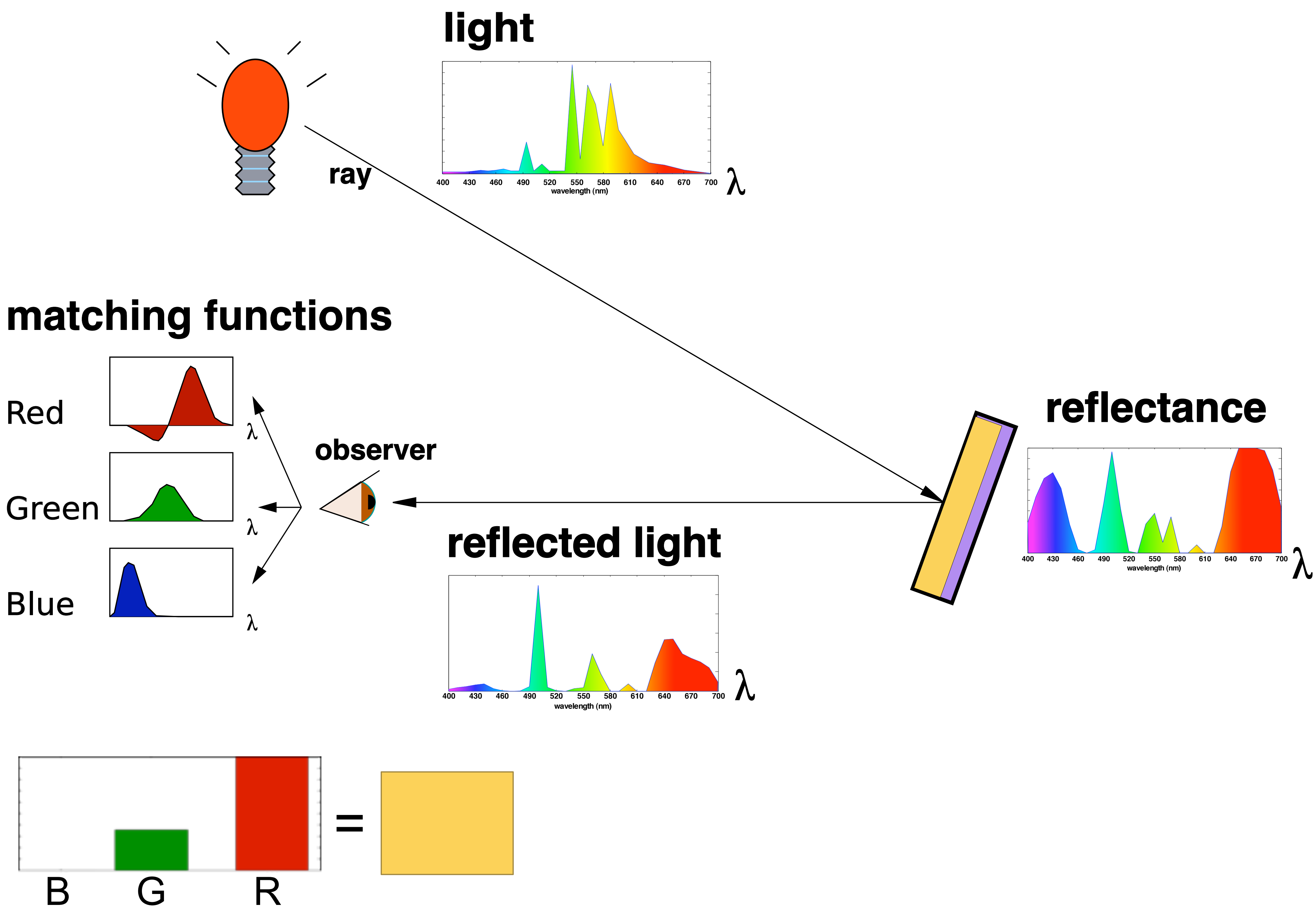

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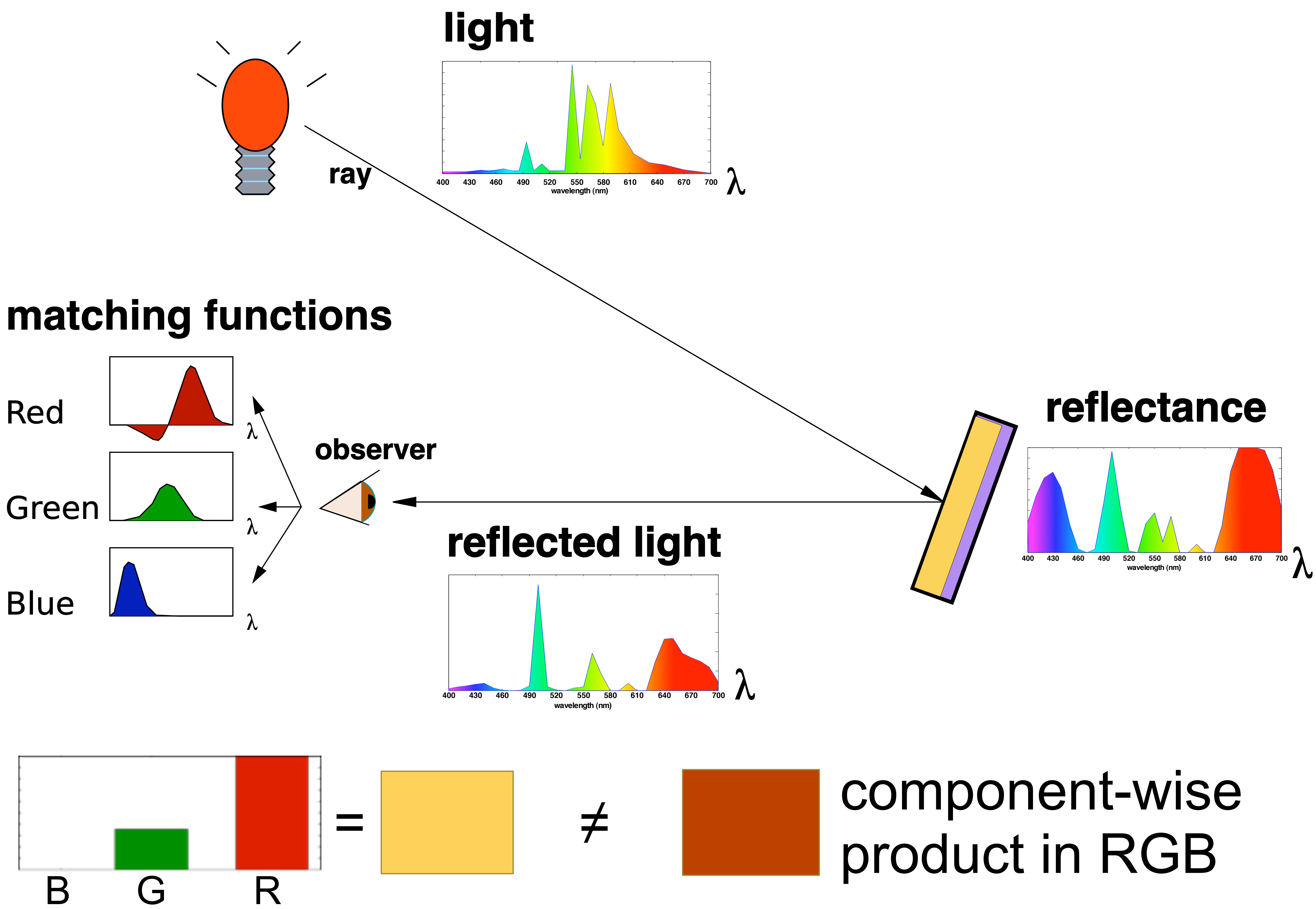
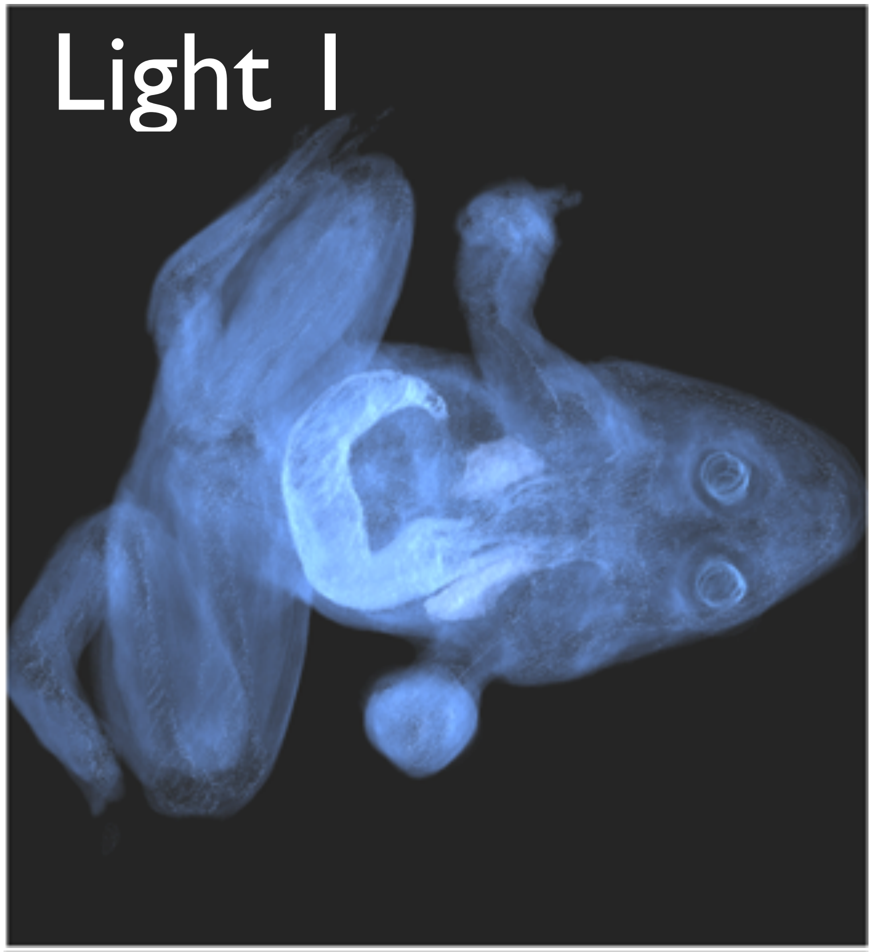


component-wise product in RGB

Use for Visualization



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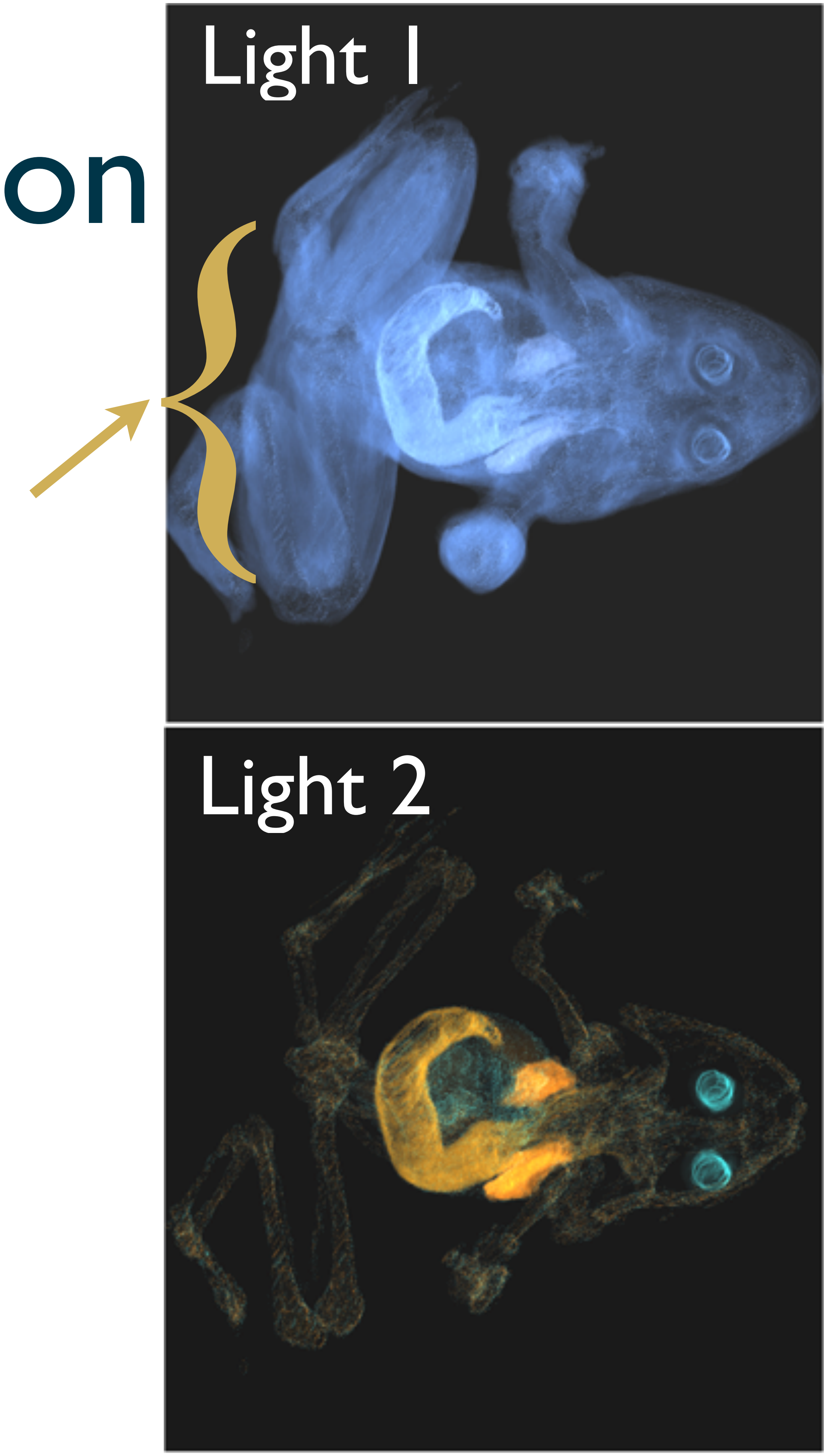
- Metamers
  - Different Spectra give same RGB



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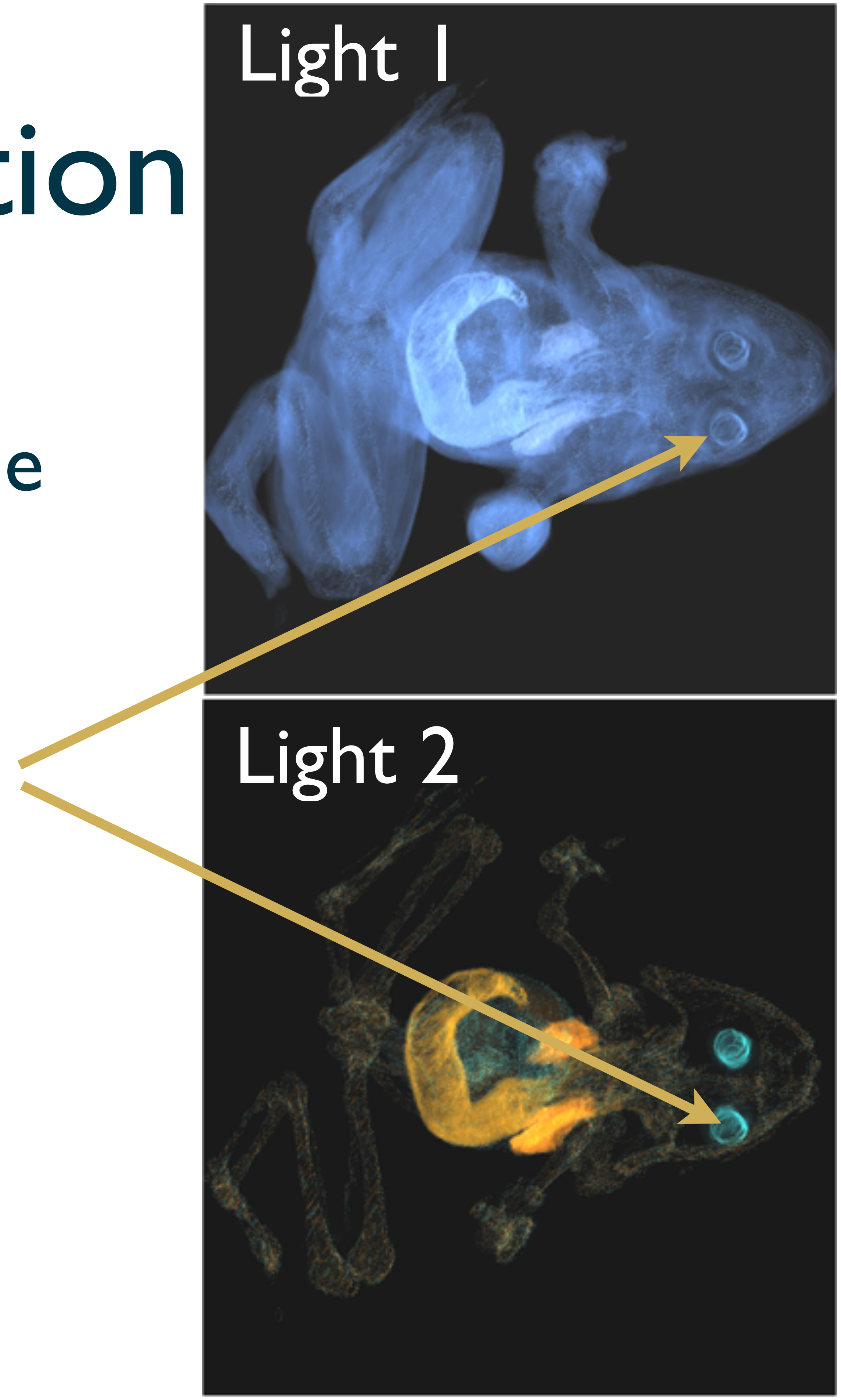
  - Metamers under changing light



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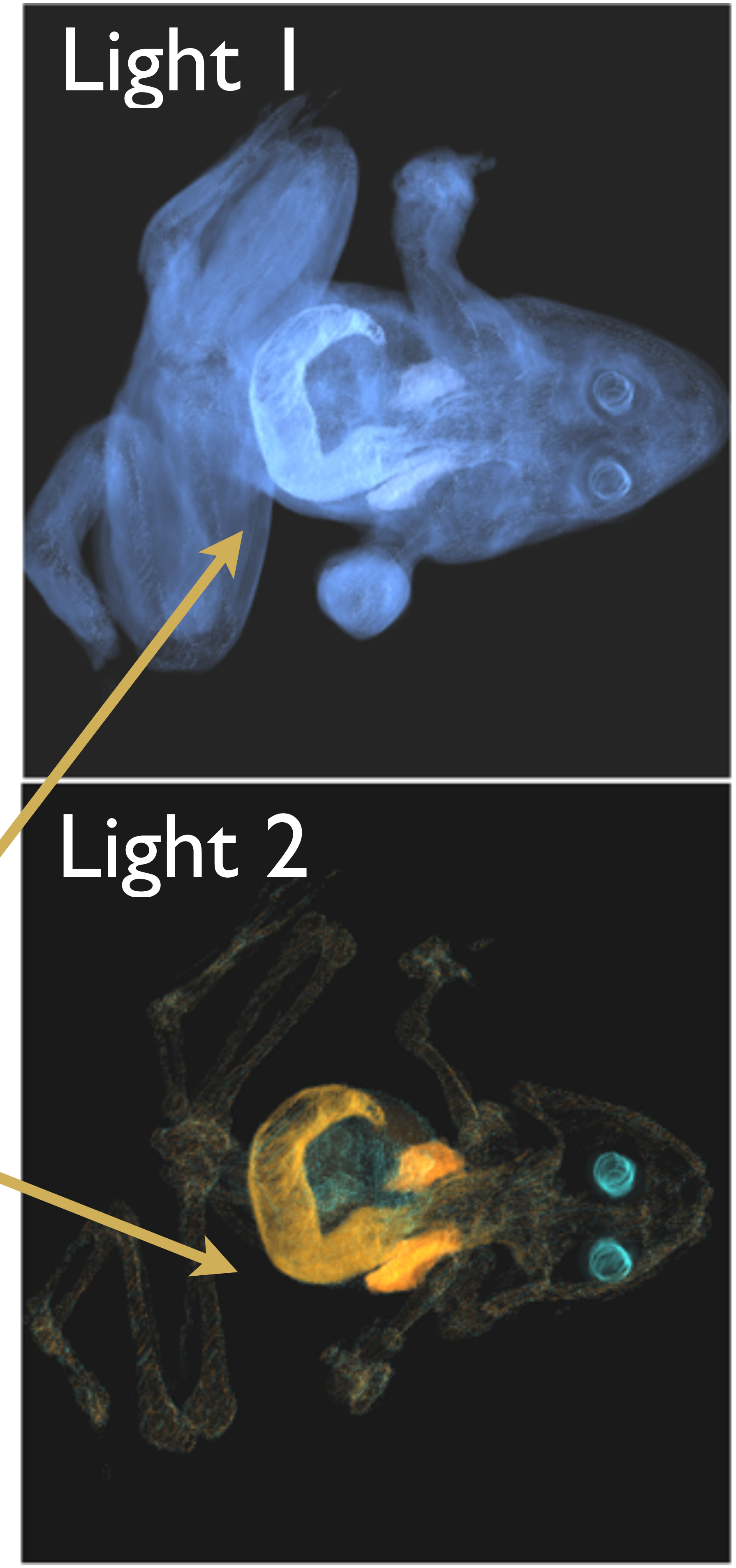

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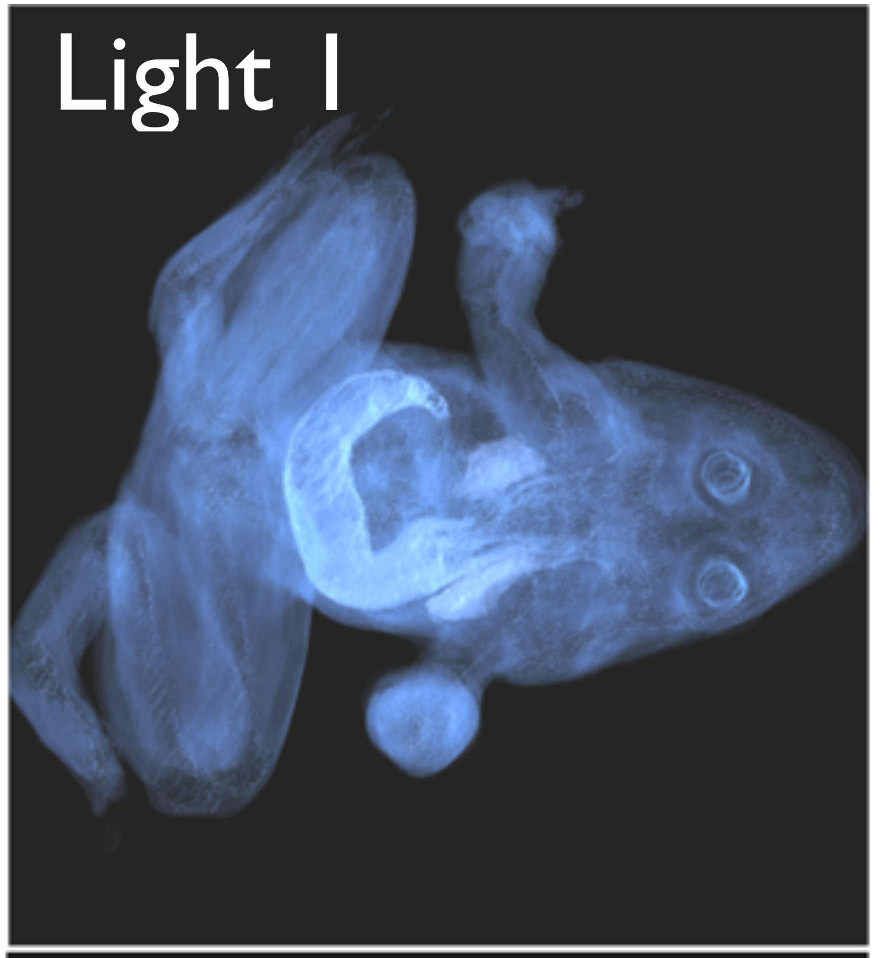
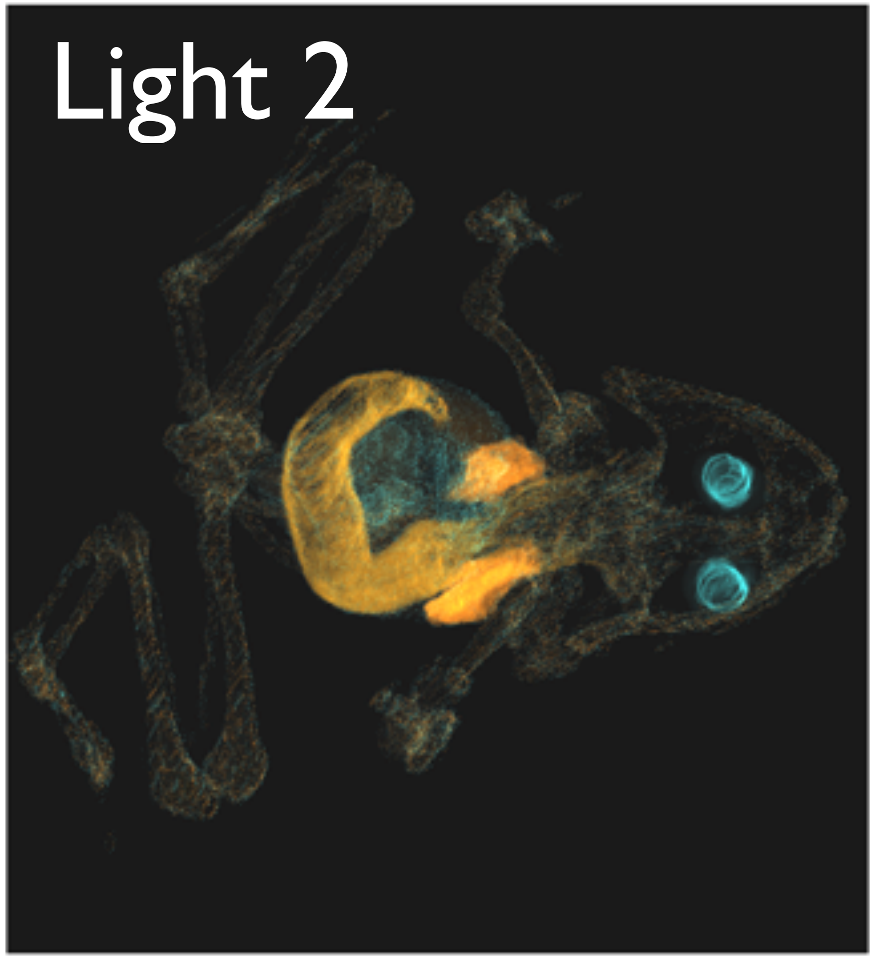
- Effective choice of light & material palette needed!





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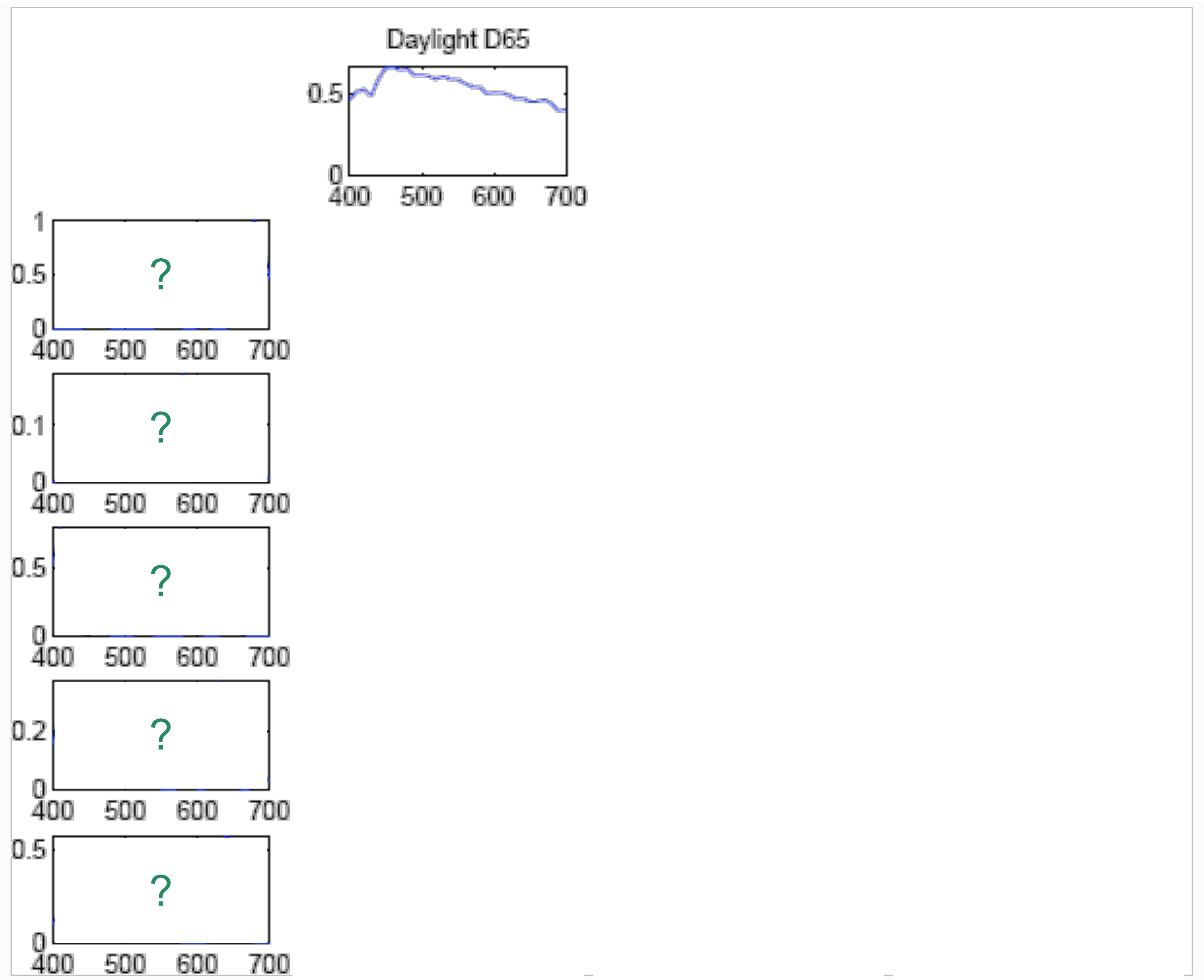

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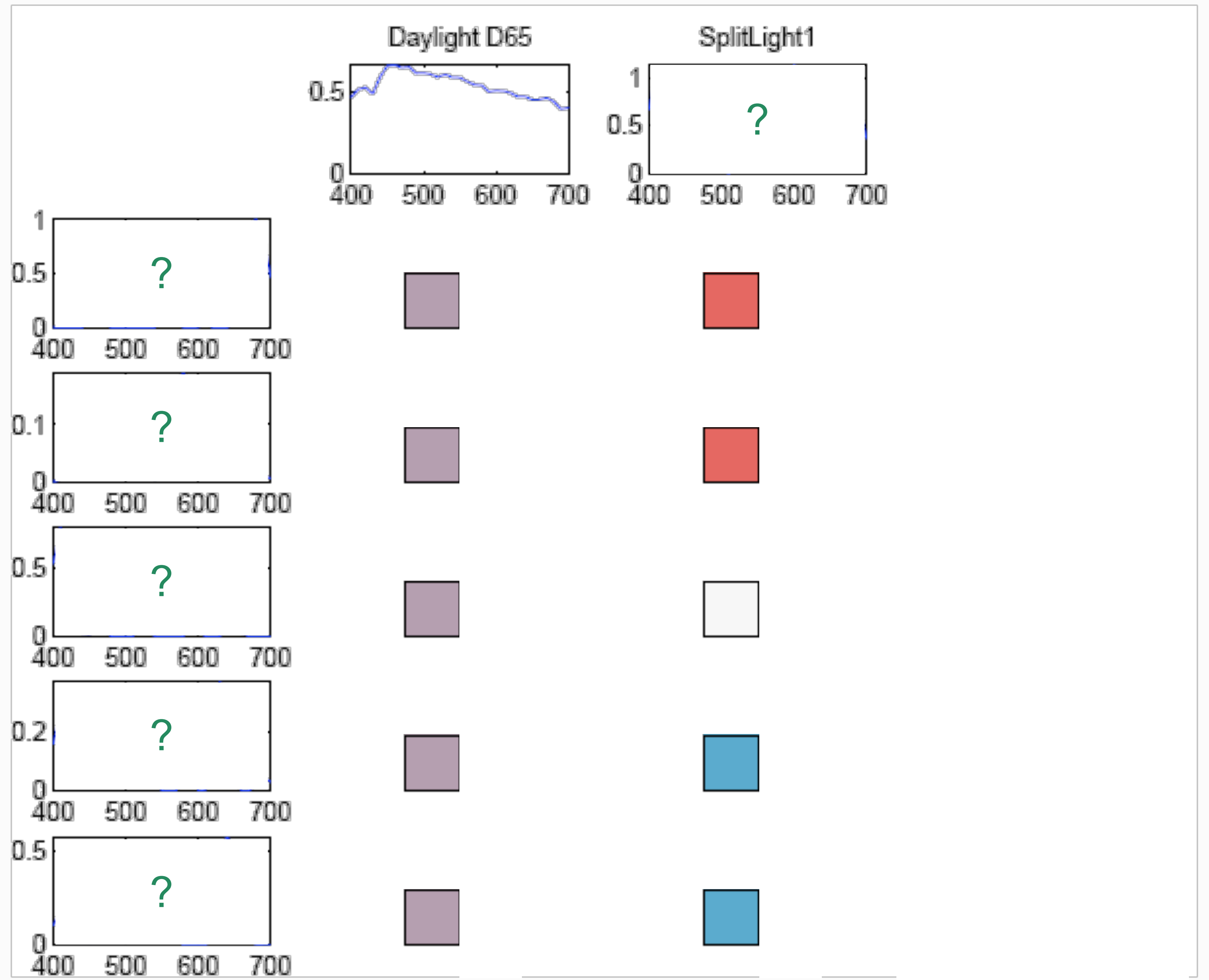

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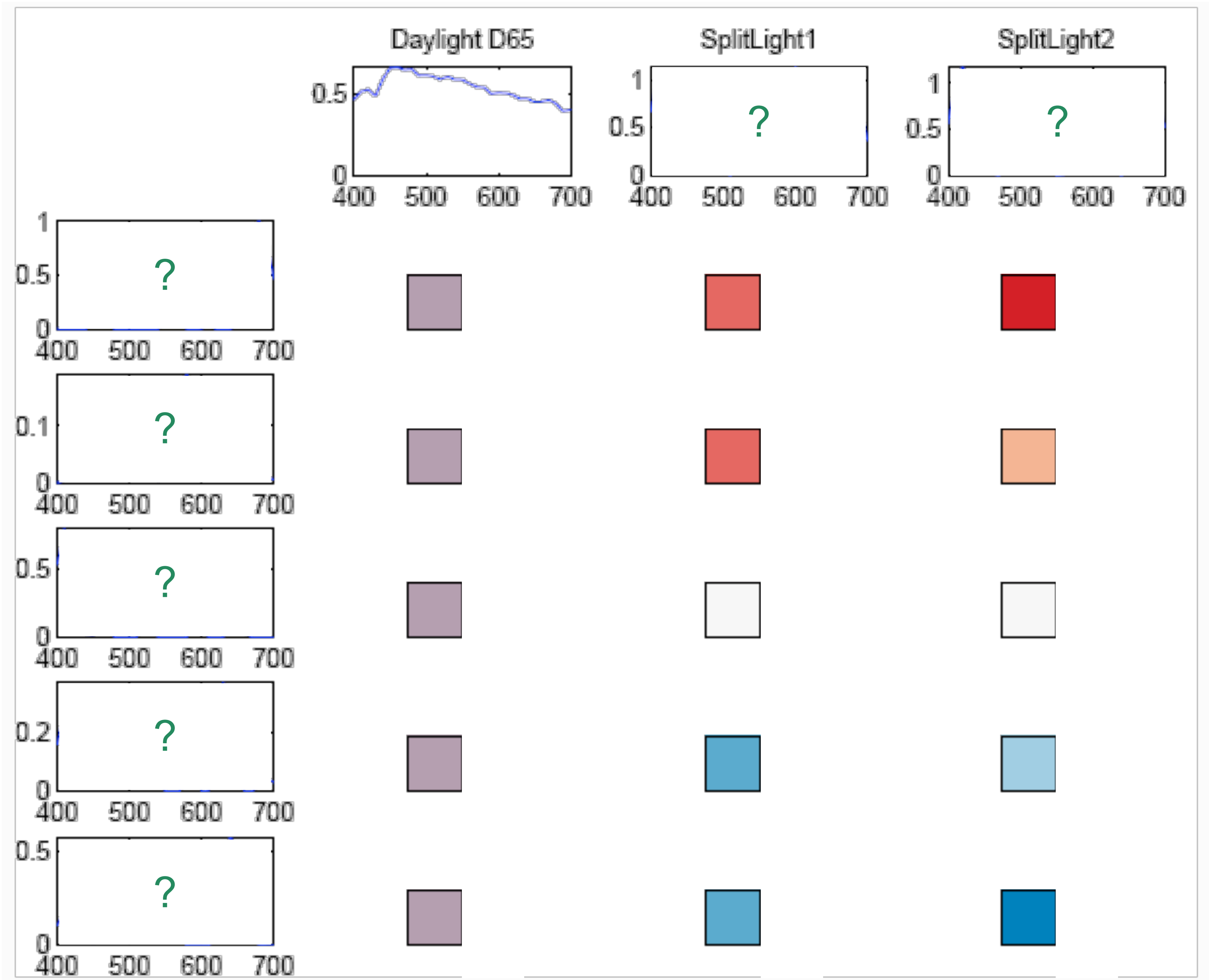

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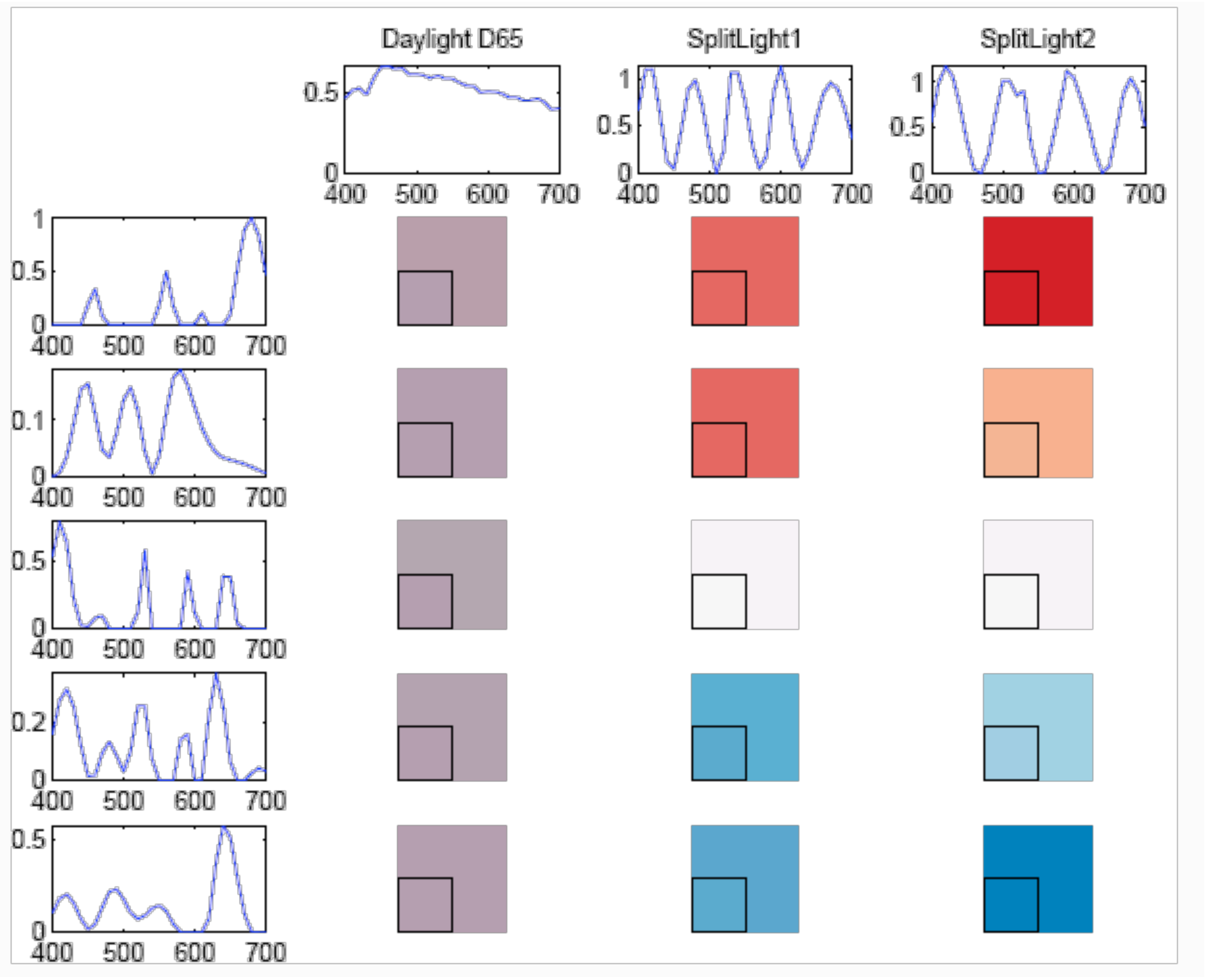

Quality Criteria

- Colour
  - Fit the desired colour or metamer
- Smoothness
  - Regularize solution and reduce extrema
- Minimal error in linear model
  - Minimal colour difference when illumination bounce is computed in linear subspace
- Positivity
  - Produce physically plausible spectra

Quality Criteria

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- Instead of equation system $\mathbf{M}\vec{x} = \vec{y}$ for spectrum $\vec{x}$
  Solve normal equation $\mathrm{argmin}_{\vec{x}} ||\mathbf{M}\vec{x} - \vec{y}||$

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  - Smoothness: $$\text{argmin}_{\vec{x}} \left\| \begin{bmatrix} -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ & & & \ddots & & \\ 0 & 0 & \cdots & -1 & 2 & -1 \end{bmatrix} \vec{x} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\|$$

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- Weight the criteria and combine as stacked matrix
  - Global minimum error solution via pseudo-inverse of $\mathbf{M}$
  - Positivity through quadratic programming

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Materials and lighting

Materials and lighting

Given: Output	Goal: Input

Materials and lighting

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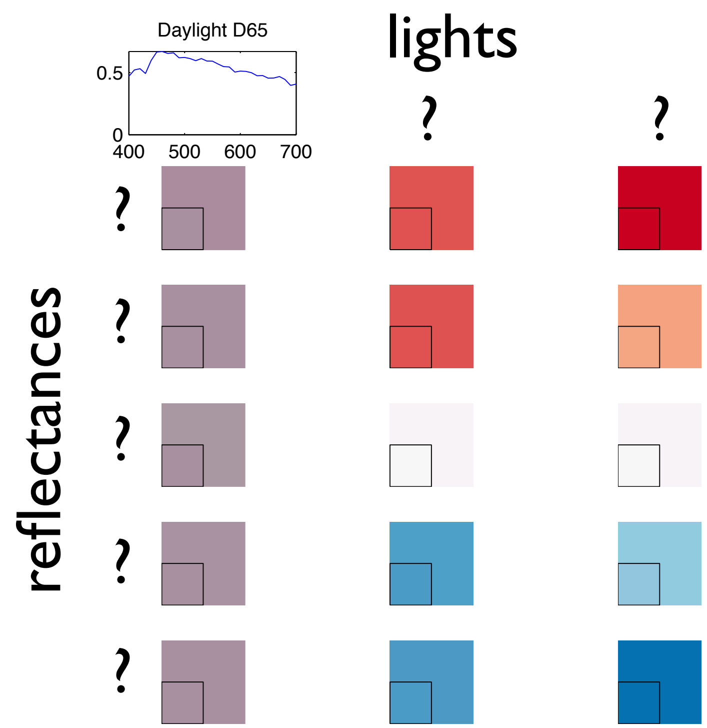

• 3x5 combination colours with 3 components each
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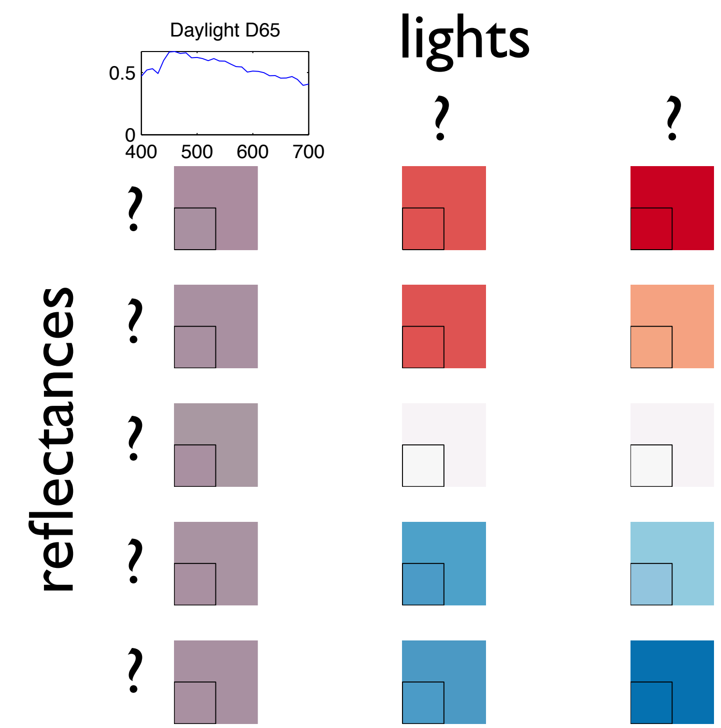

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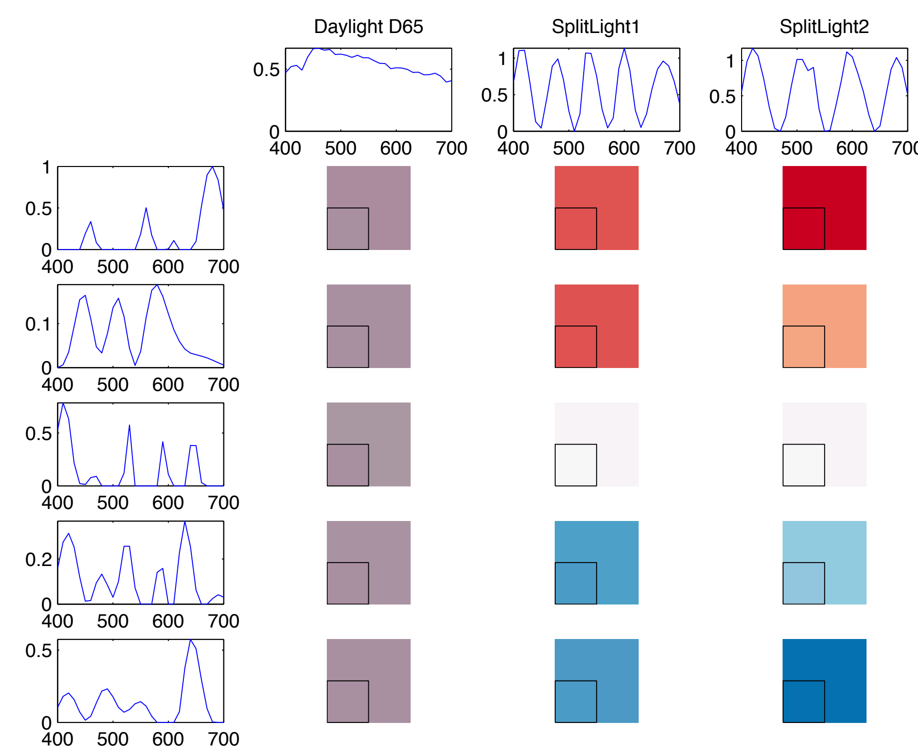
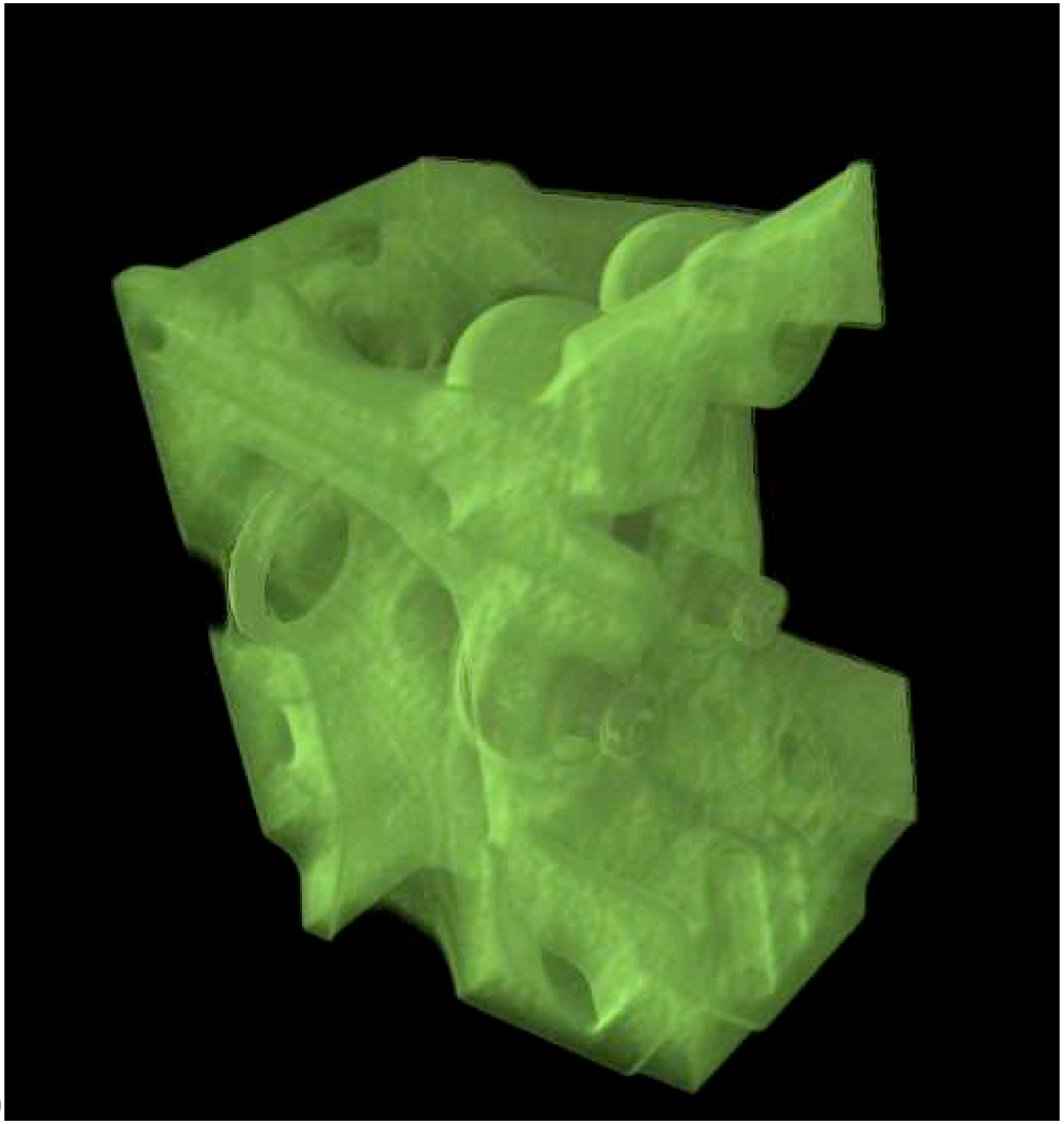


- 7x31 component SPDs

Image based re-lighting



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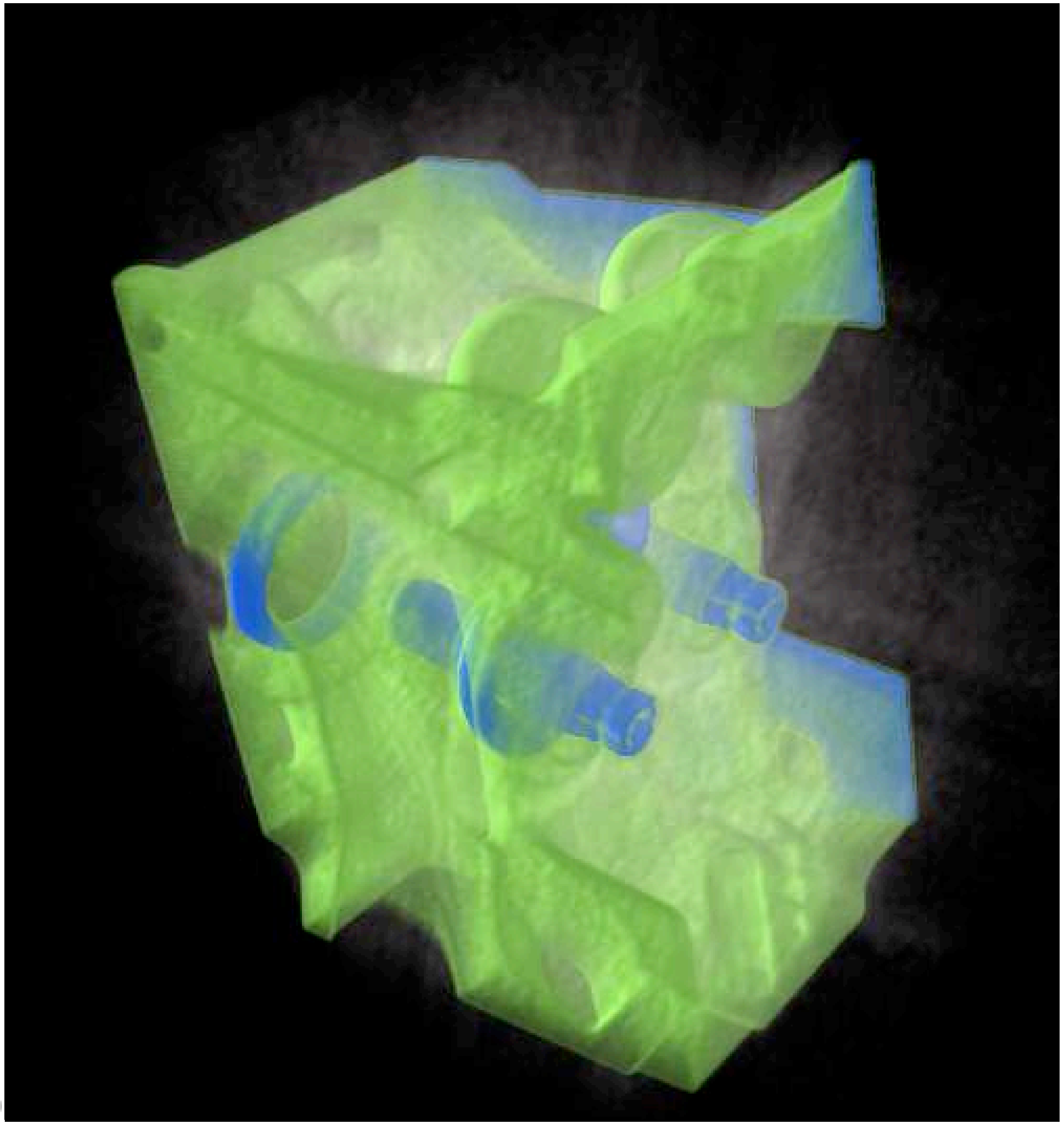

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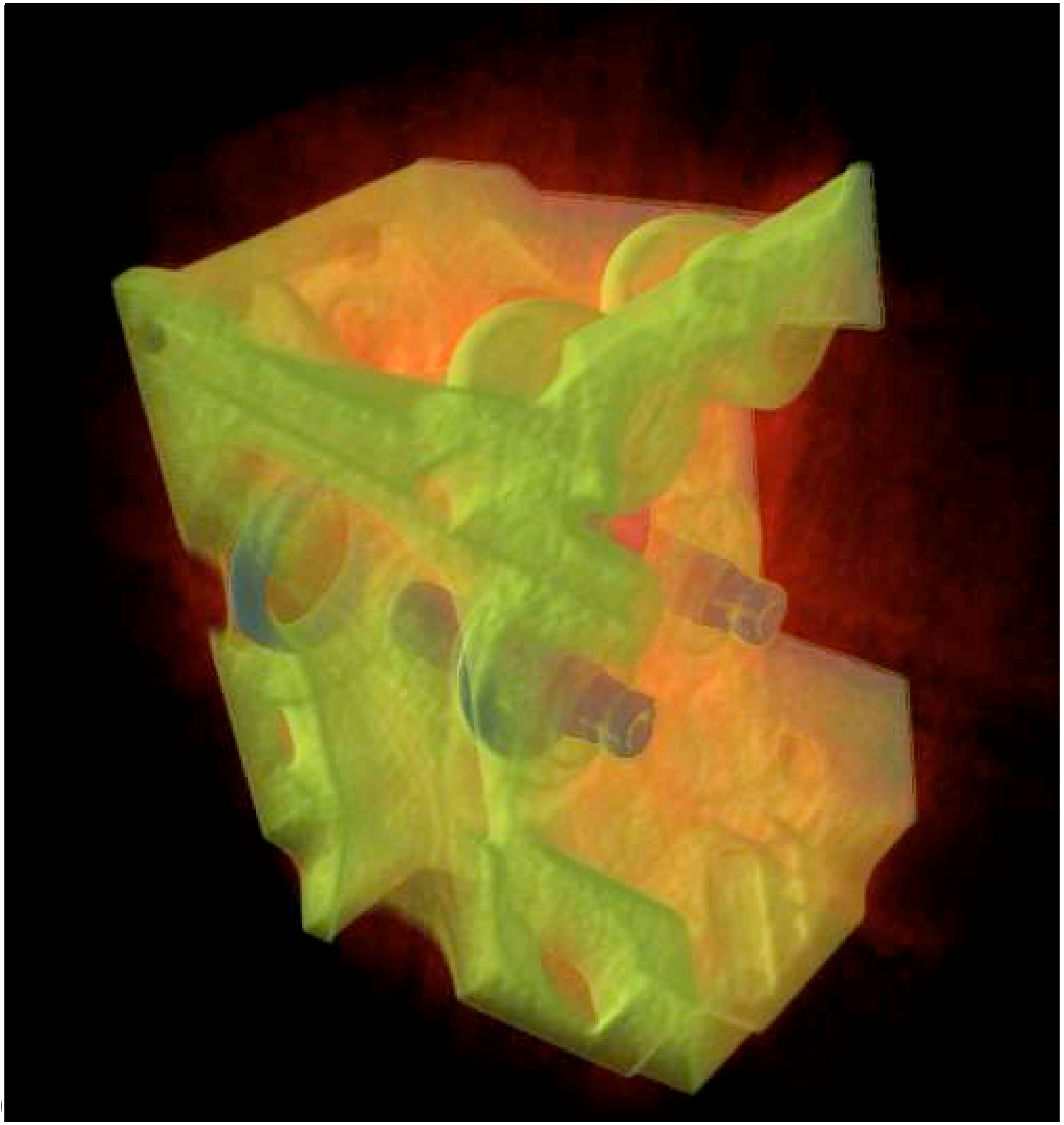
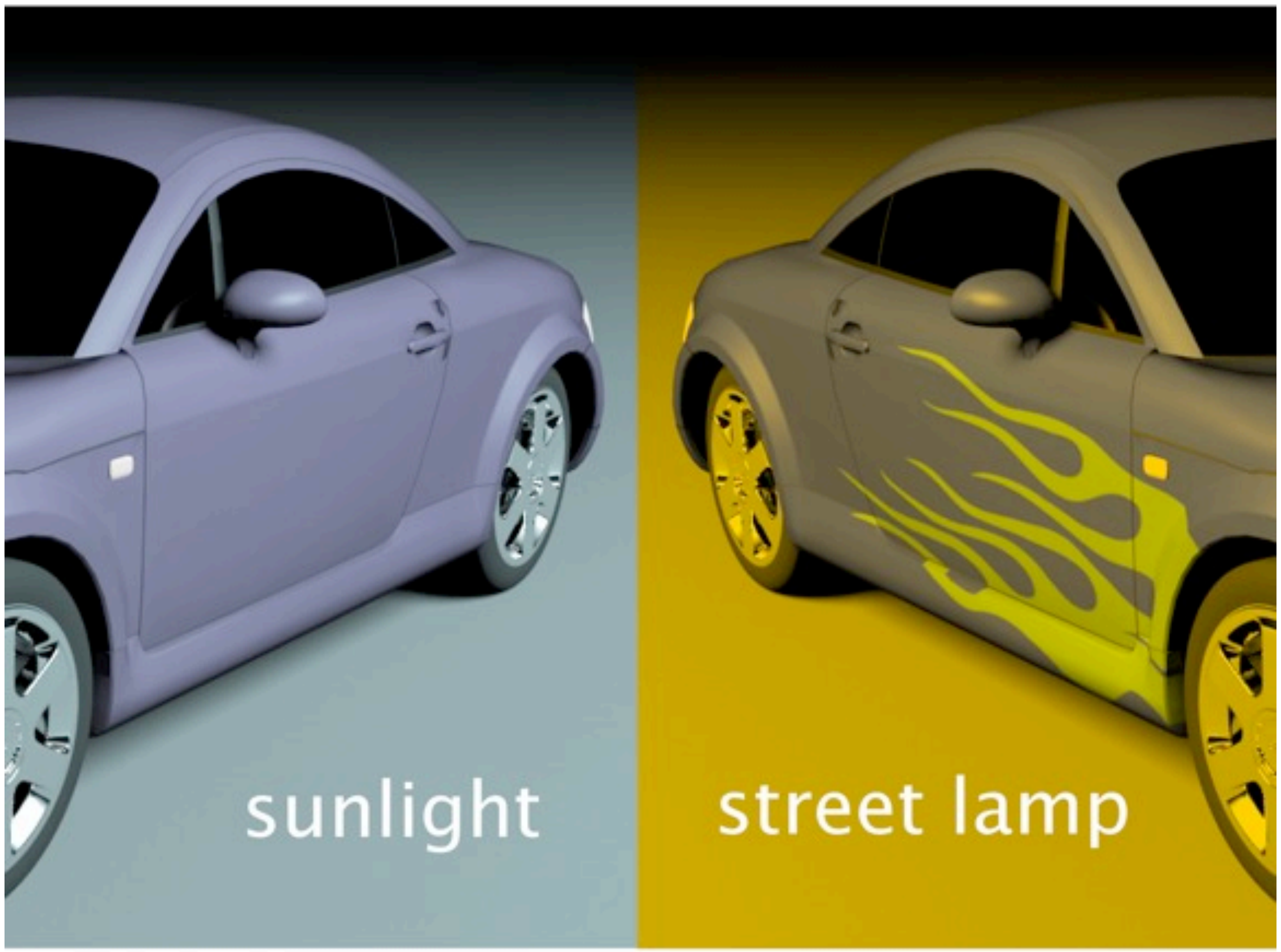
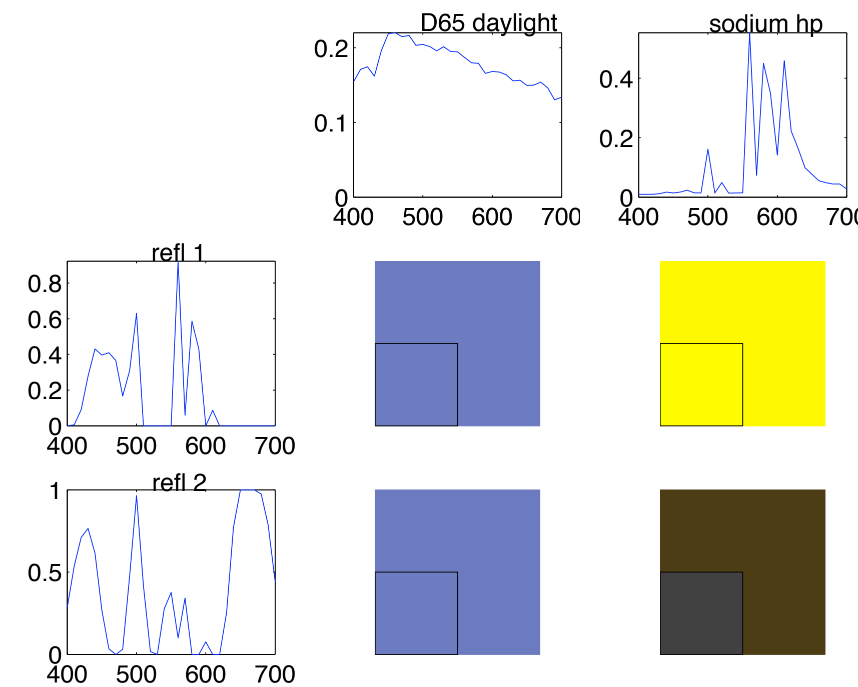


Applications in Graphics and Visualization





- Additional texture details appear under changing illumination

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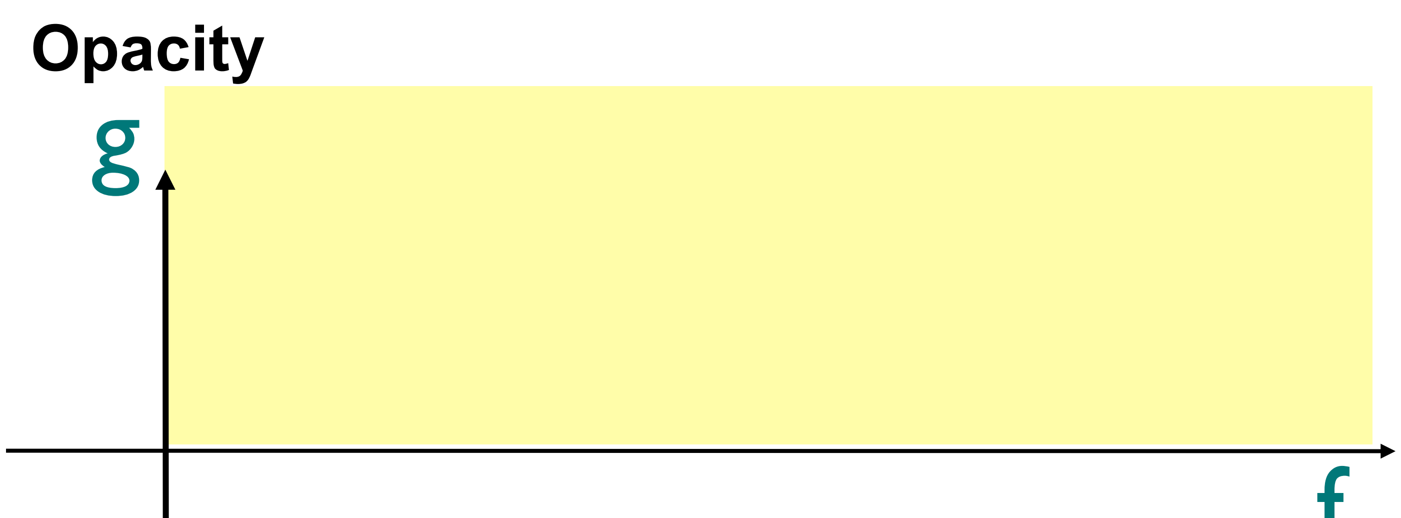

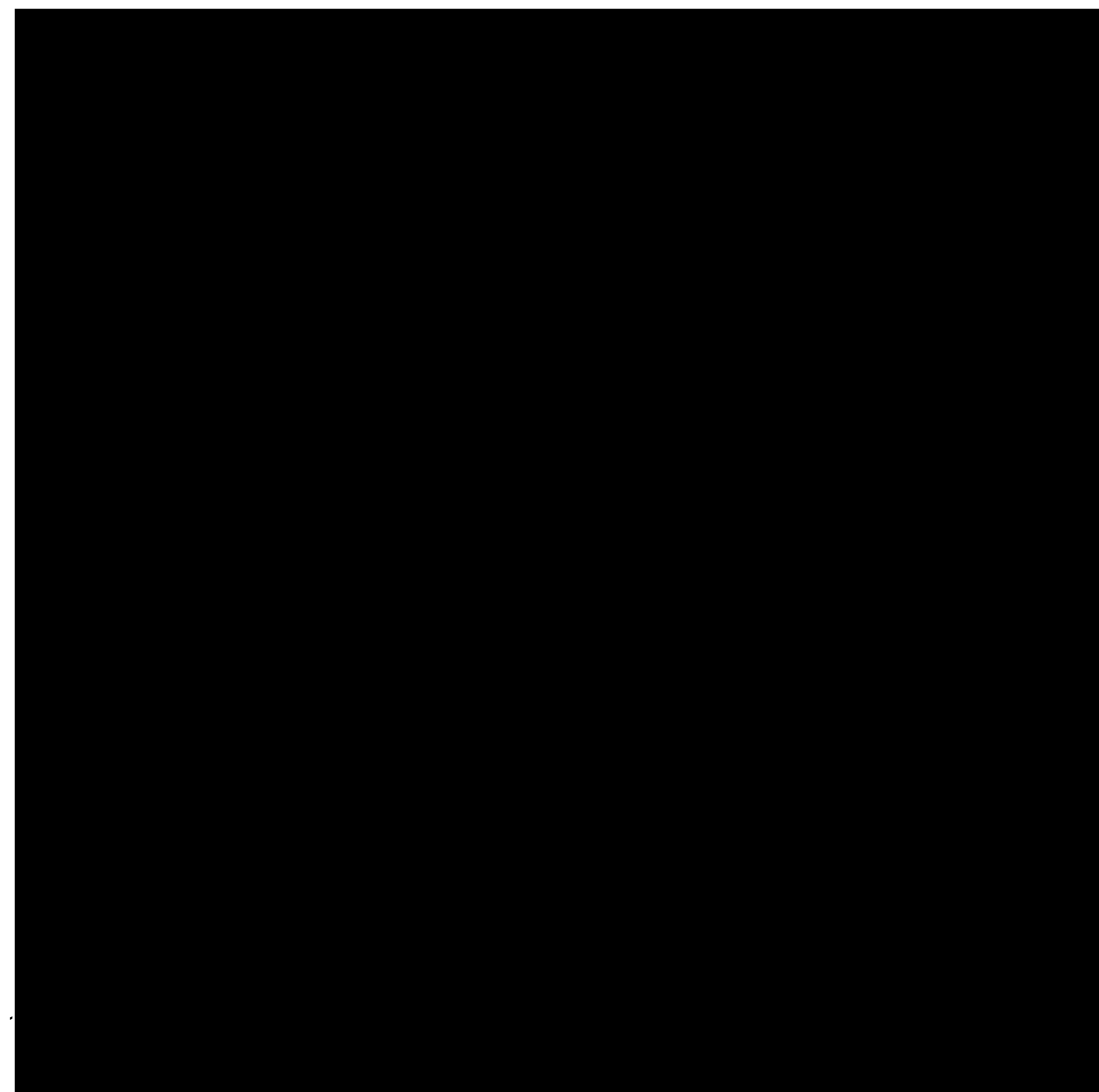

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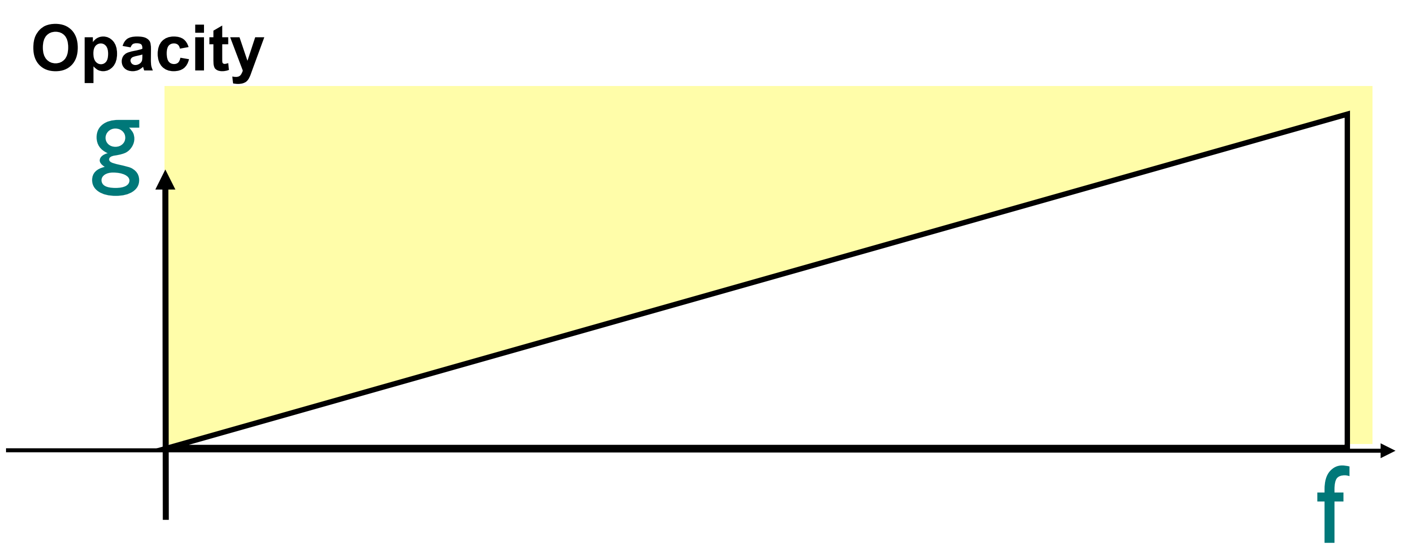

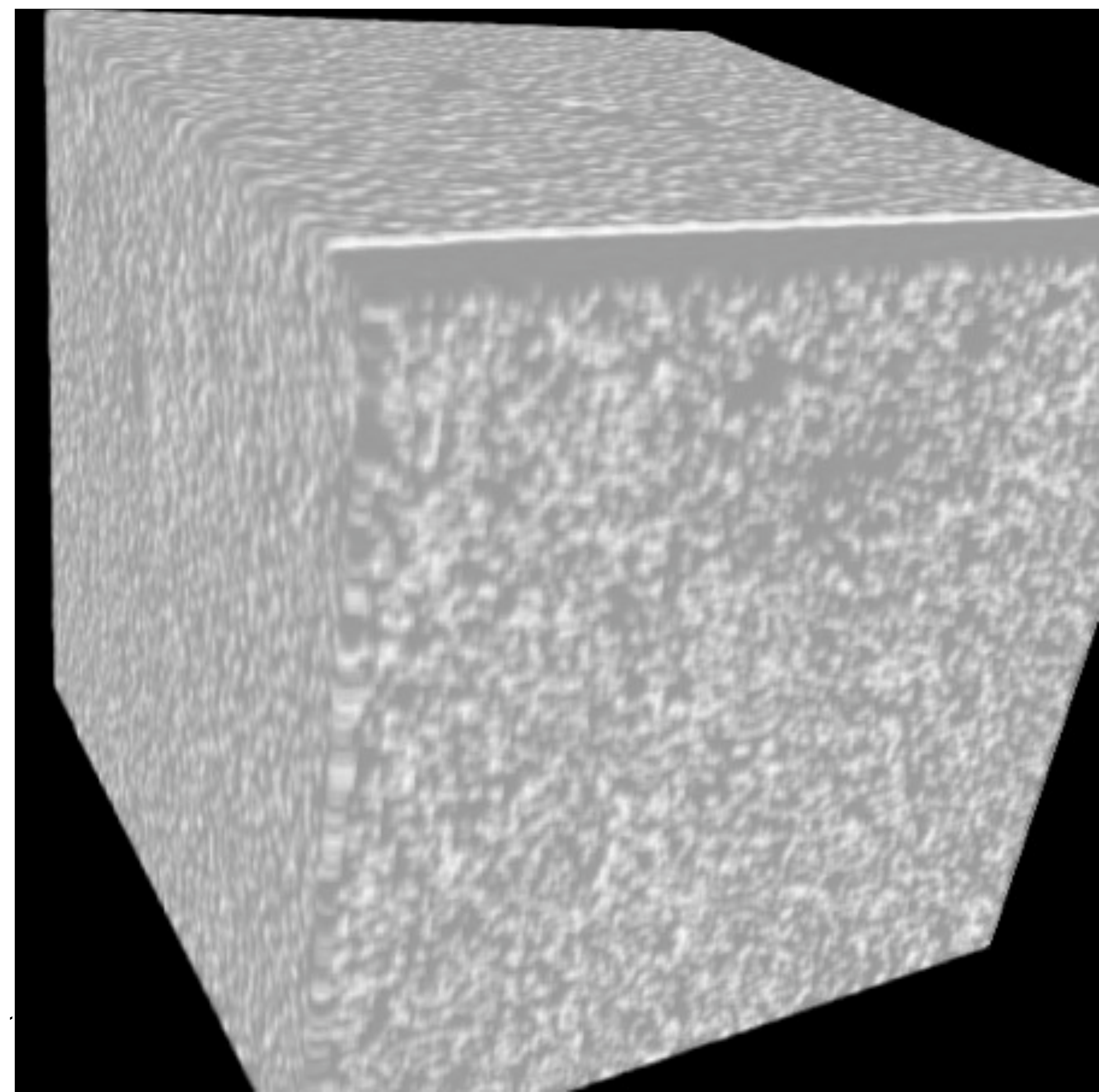

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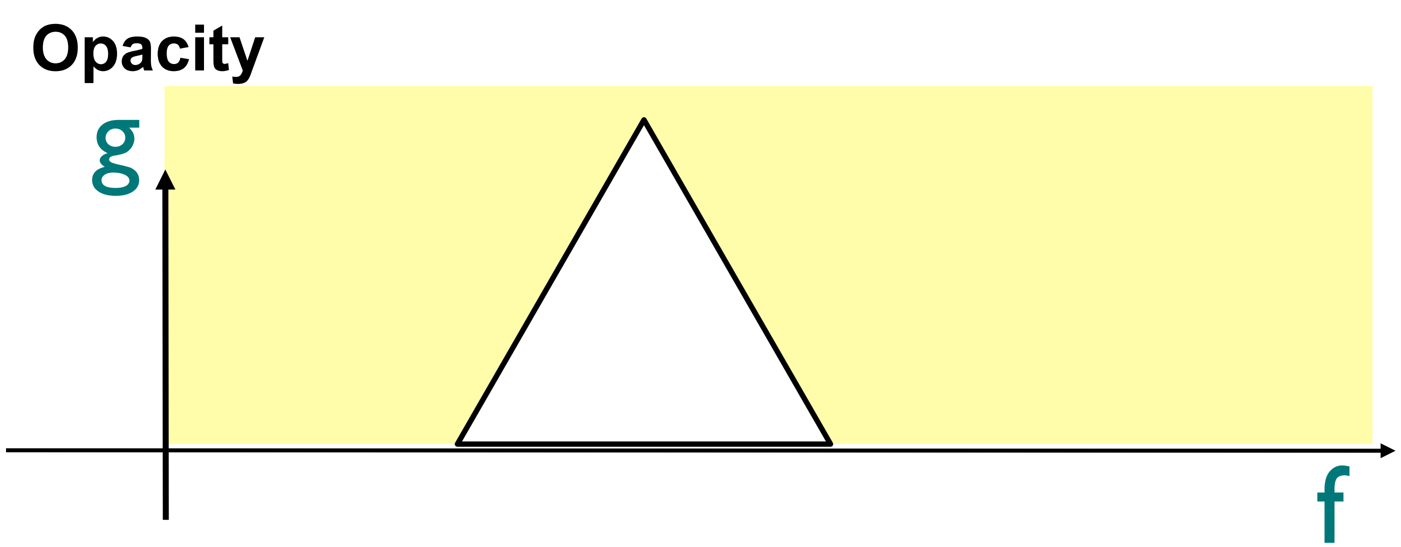

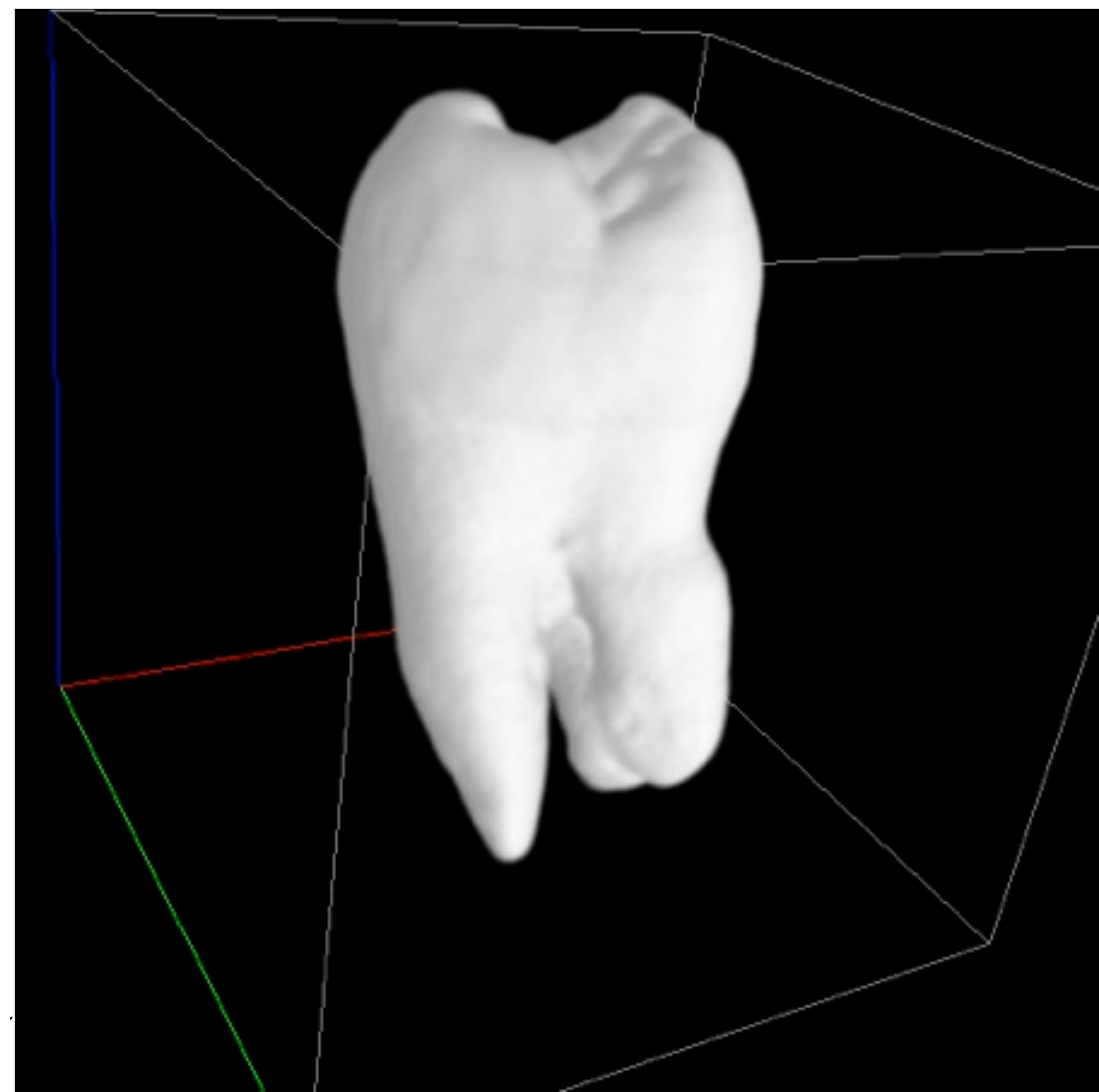

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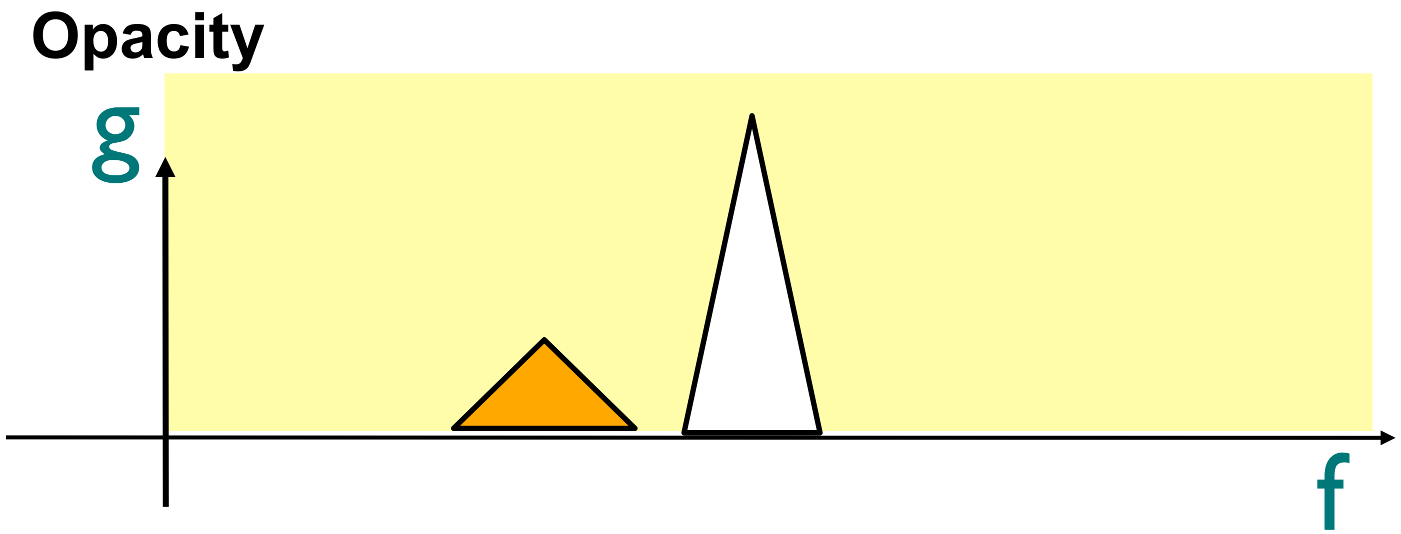

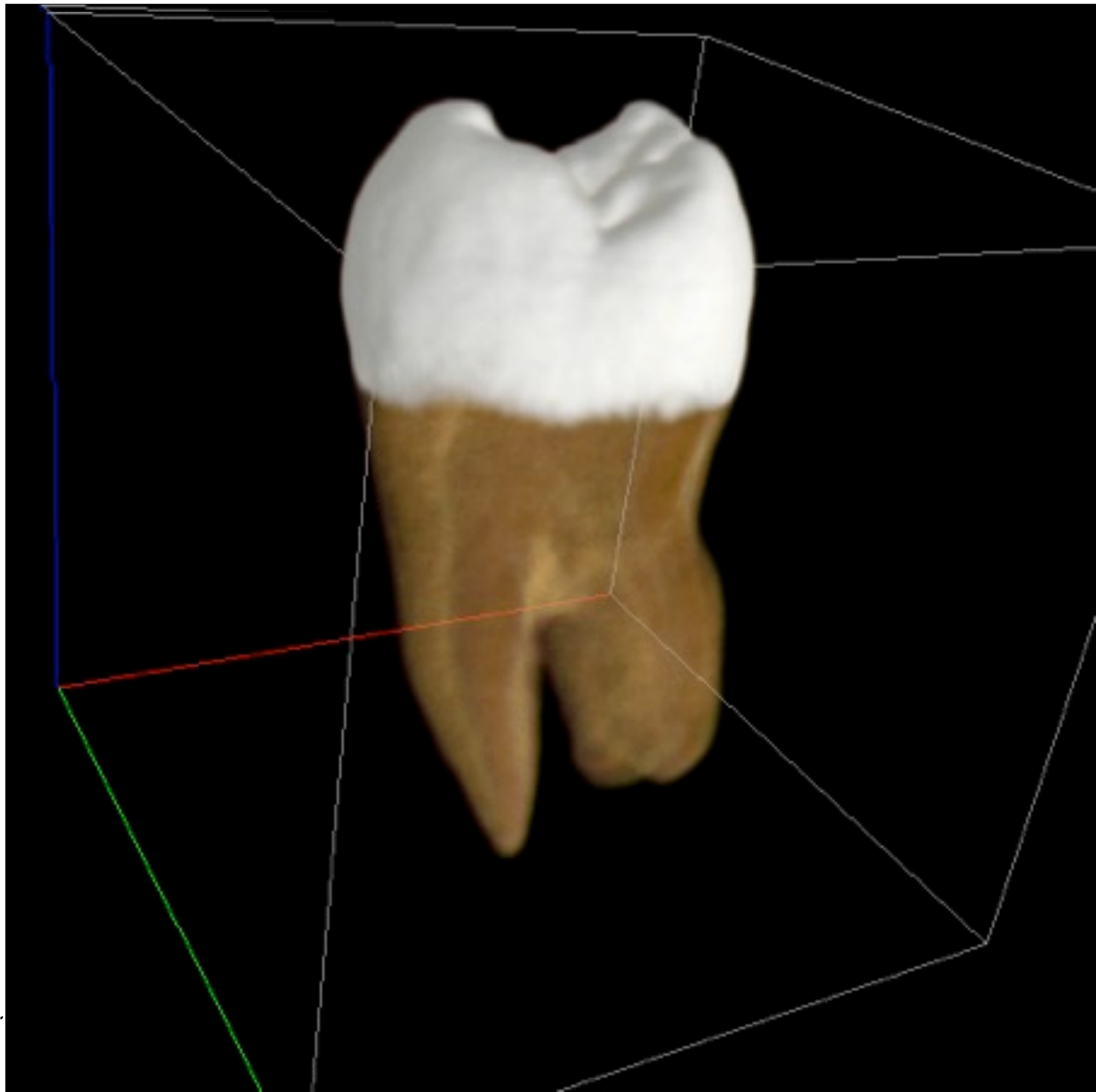

Example of g(f(x))

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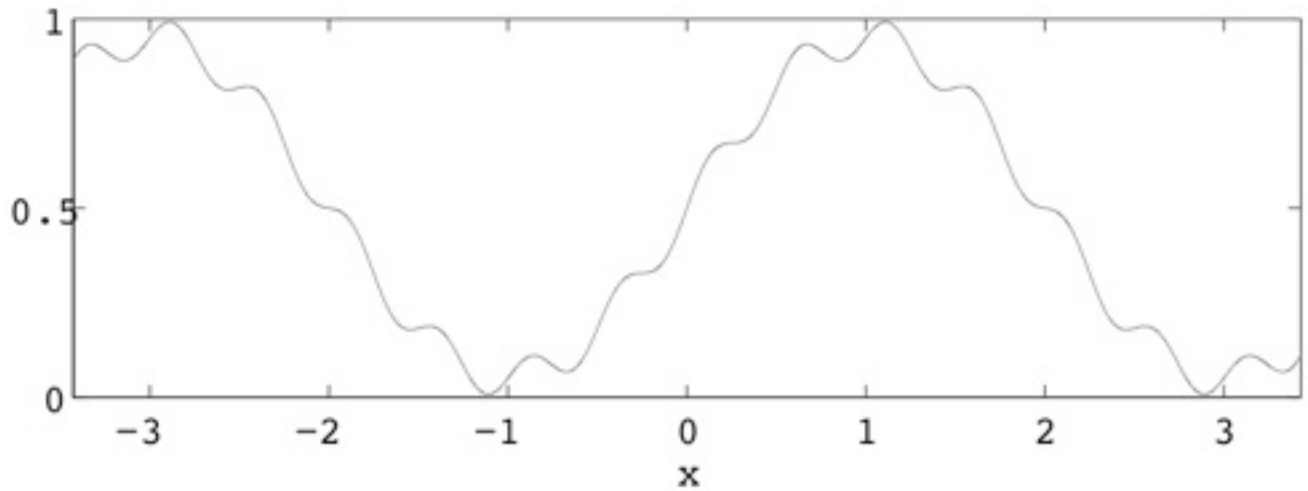

Original function f(x)

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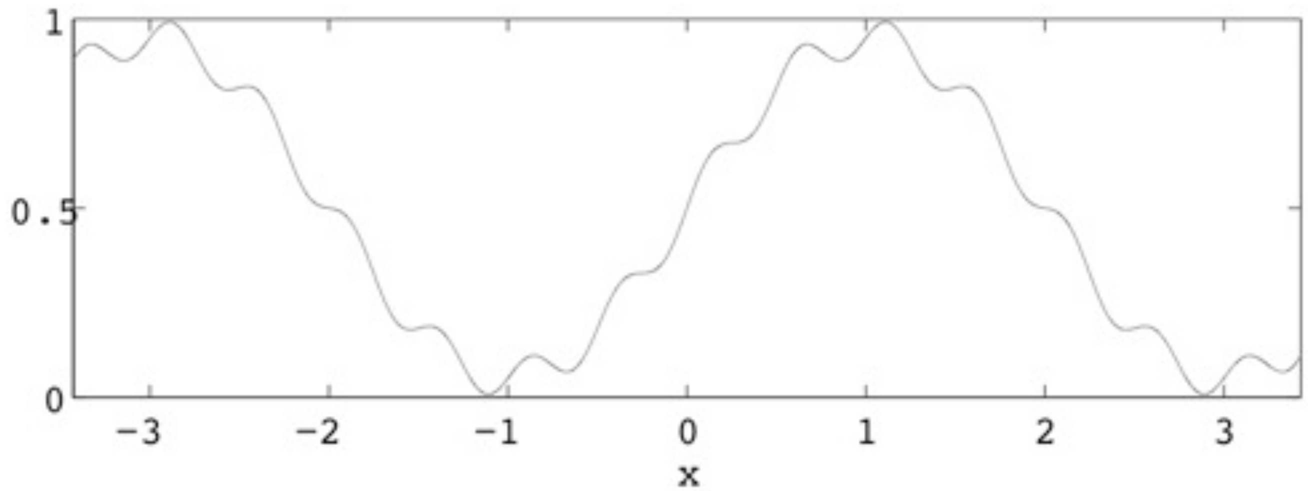

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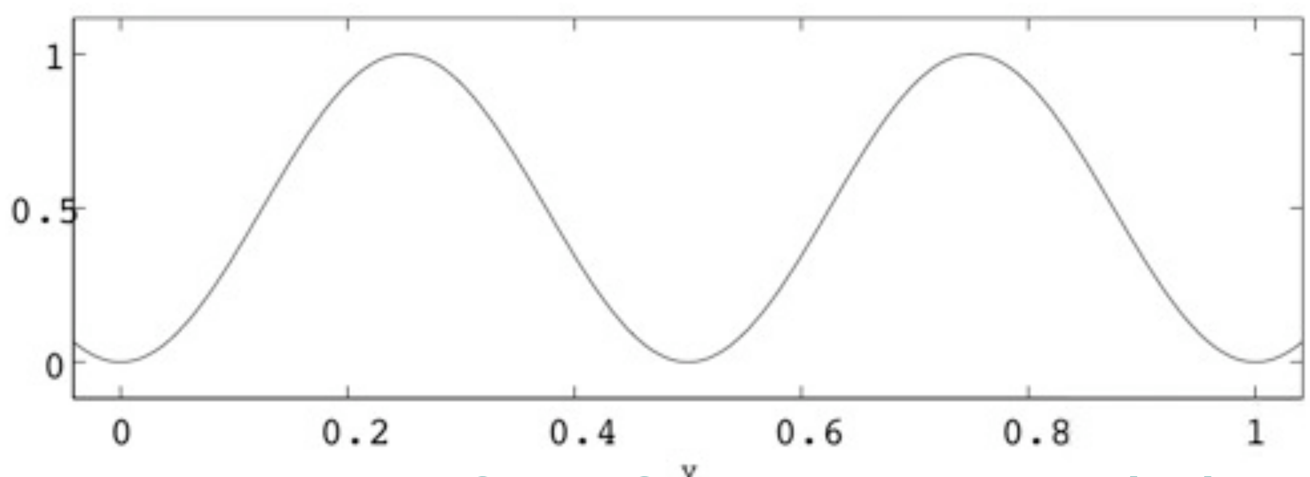

Transfer function g(y)

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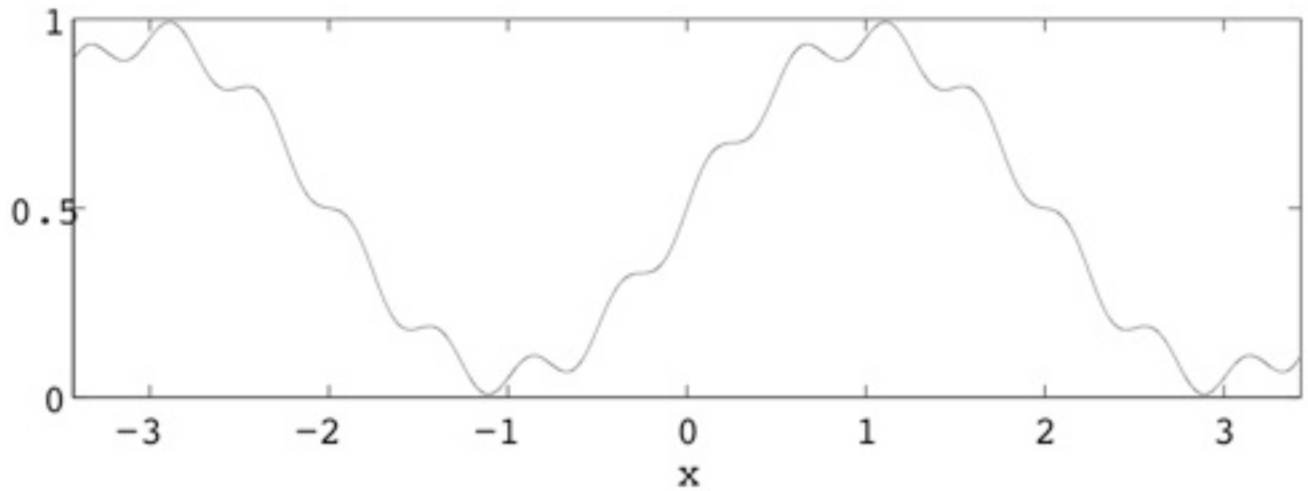

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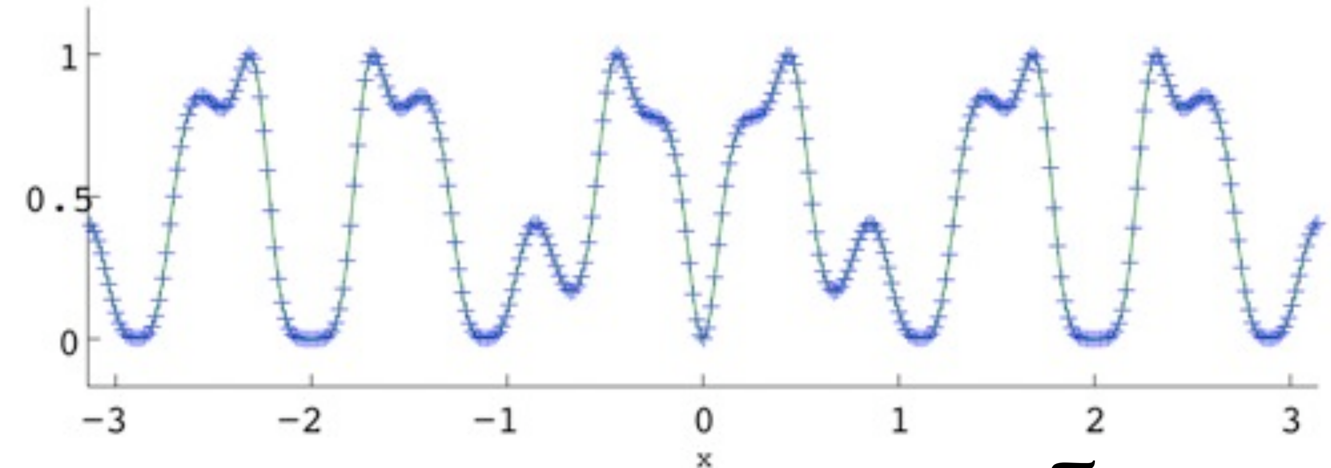
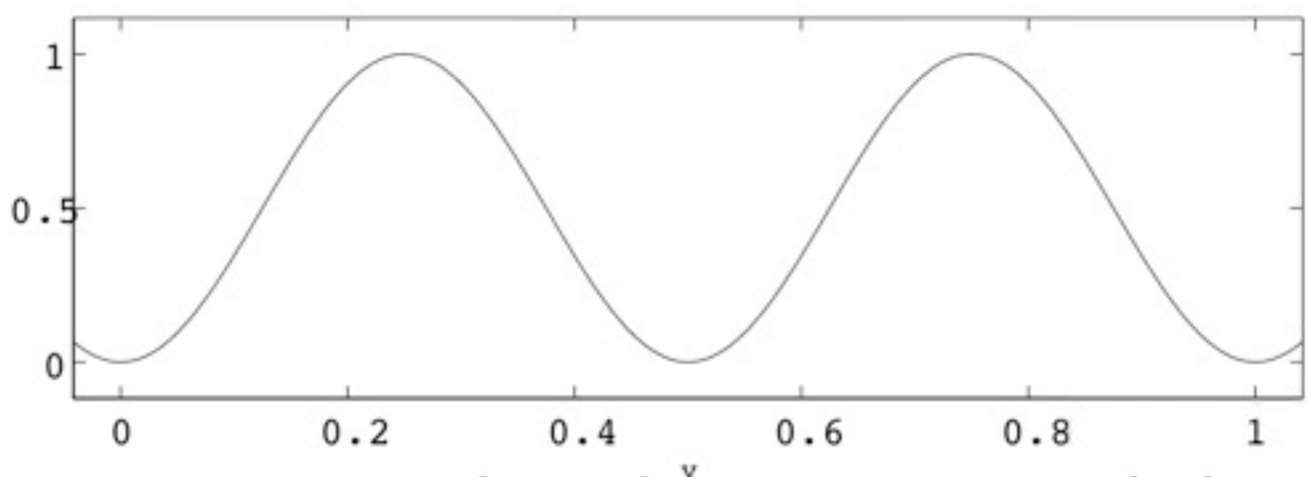


g(f(x)) sampled by $\frac{\pi}{2}\upsilon_f\upsilon_g$



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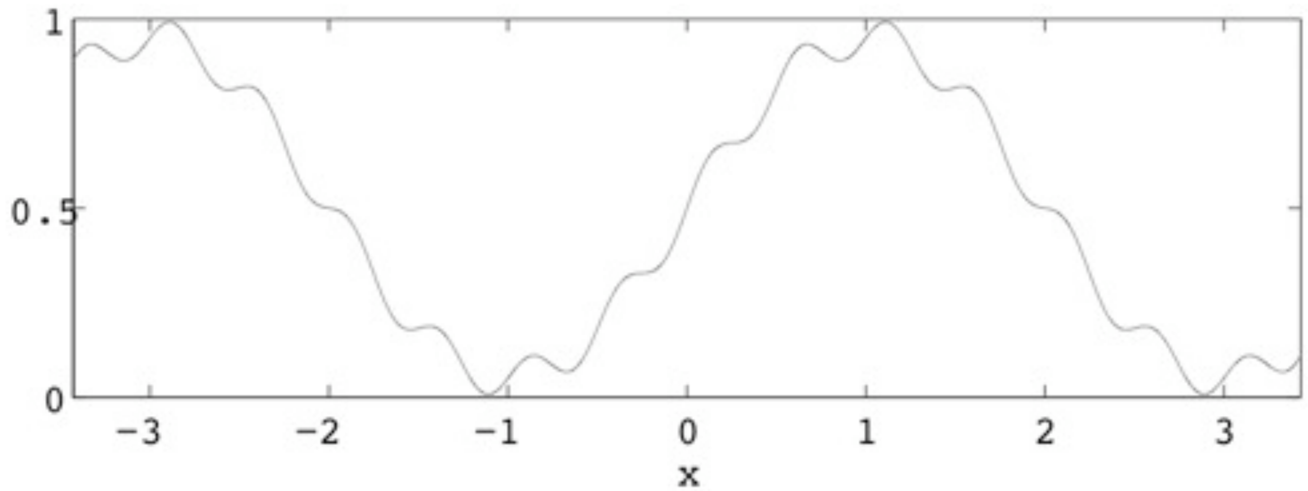

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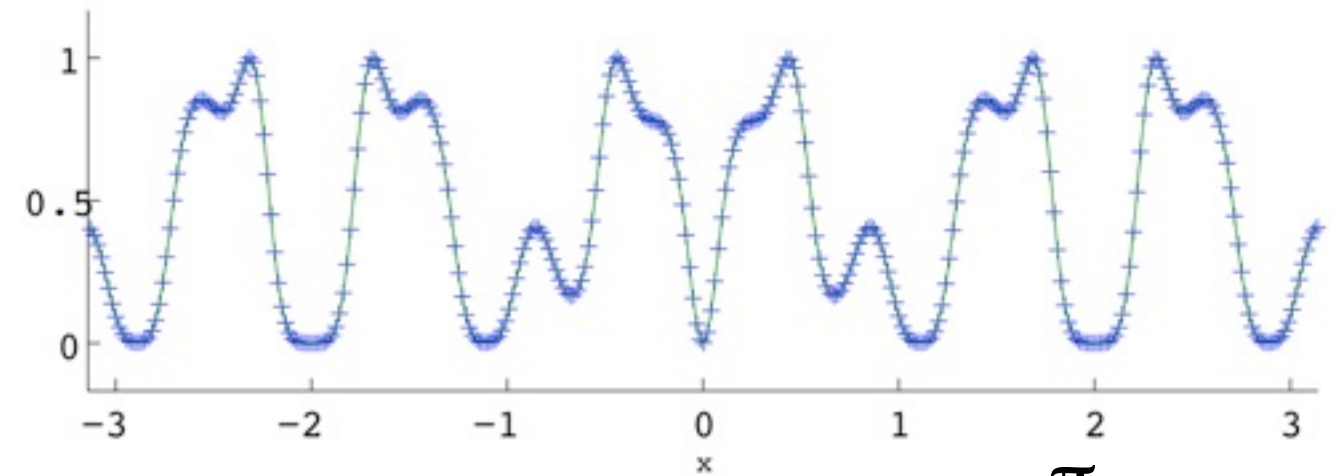

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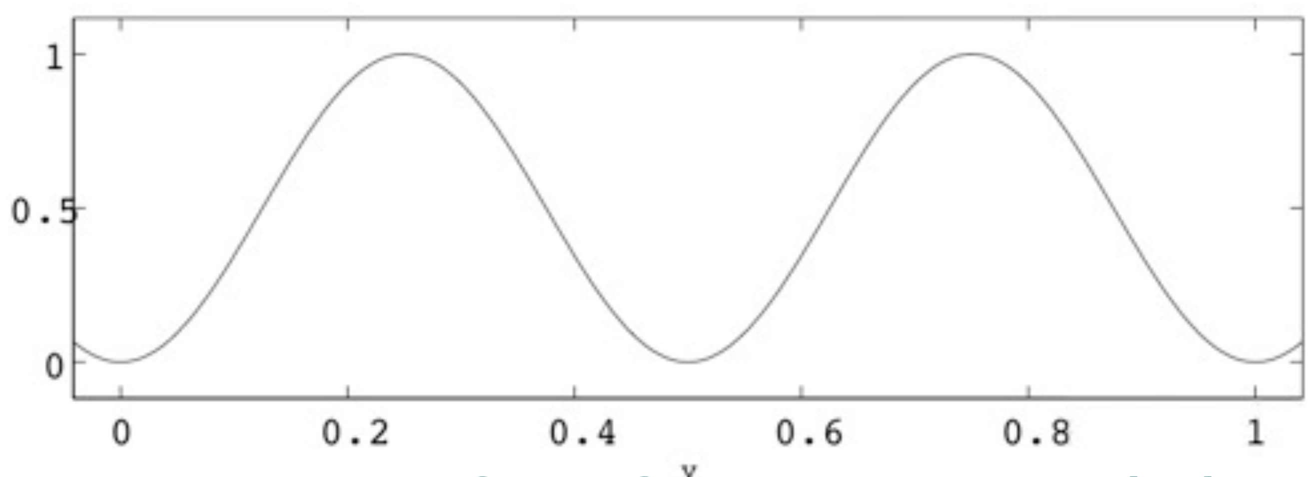

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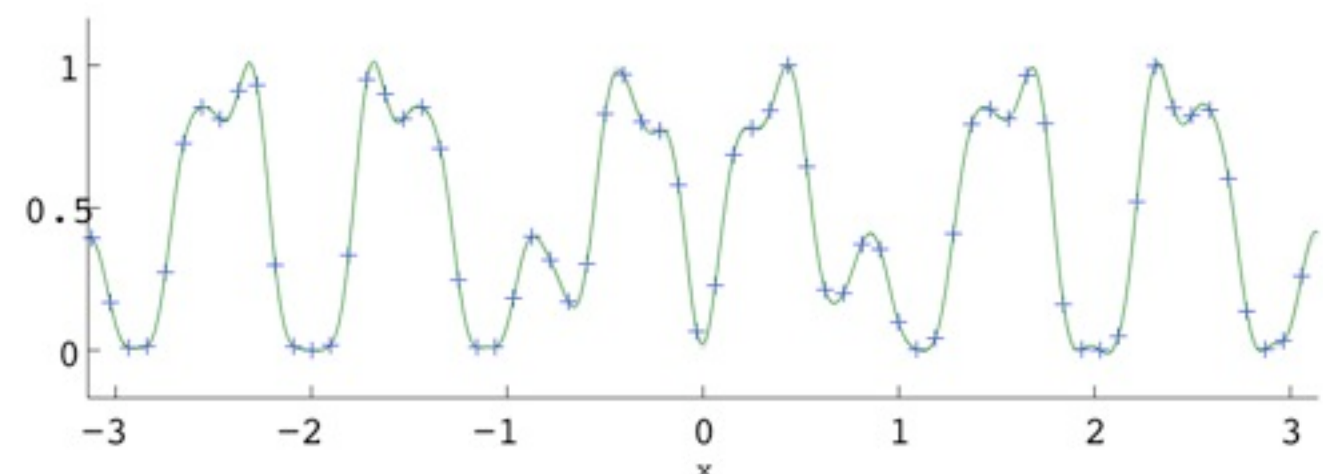

g(f(x)) sampled by $\max|f'|\upsilon_g$

Composition in Frequency Domain

$$g(\quad y \quad) = \frac{1}{\sqrt{2\pi}} \int_R G(l) e^{il \,\cdot\, y} \; dl$$

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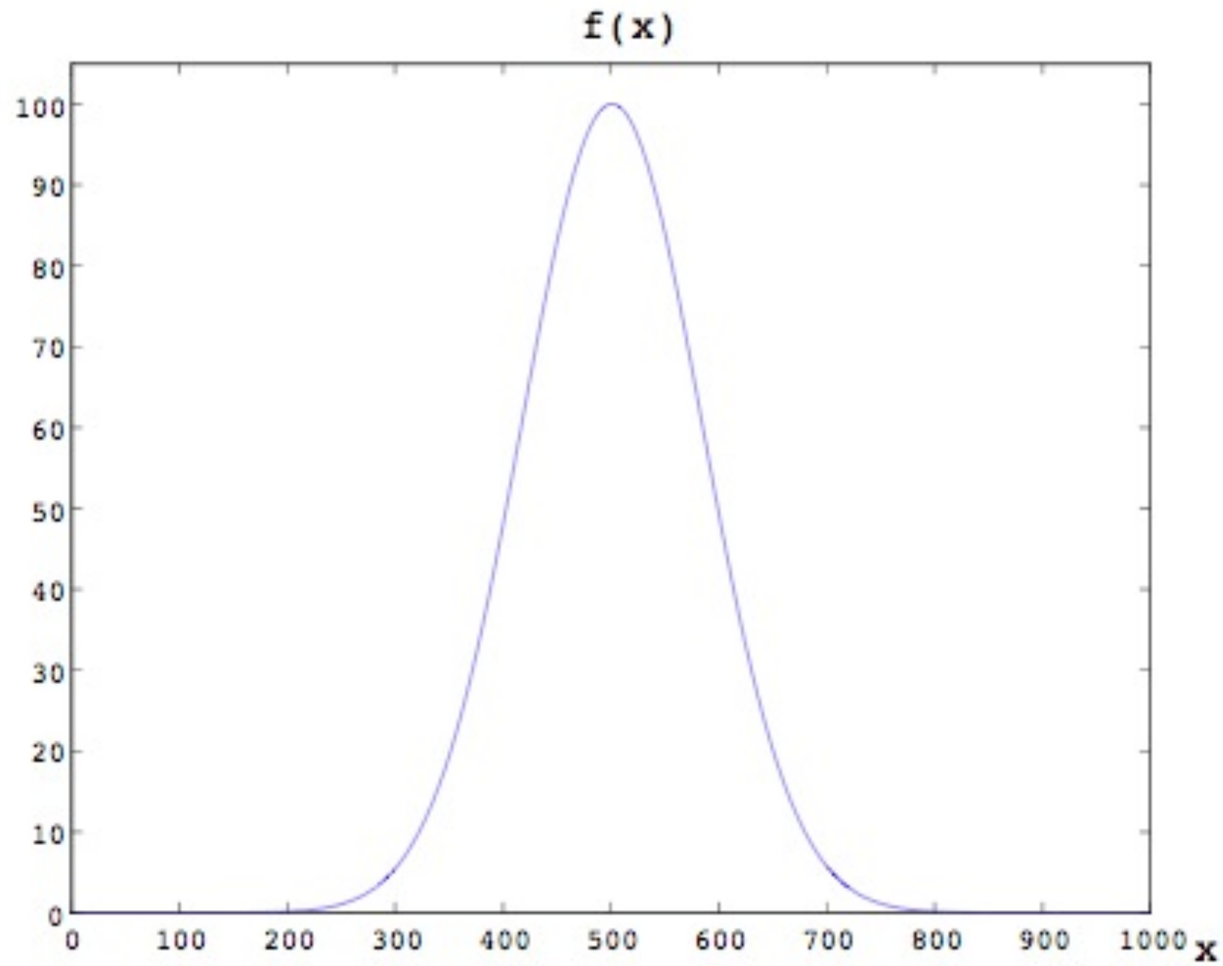

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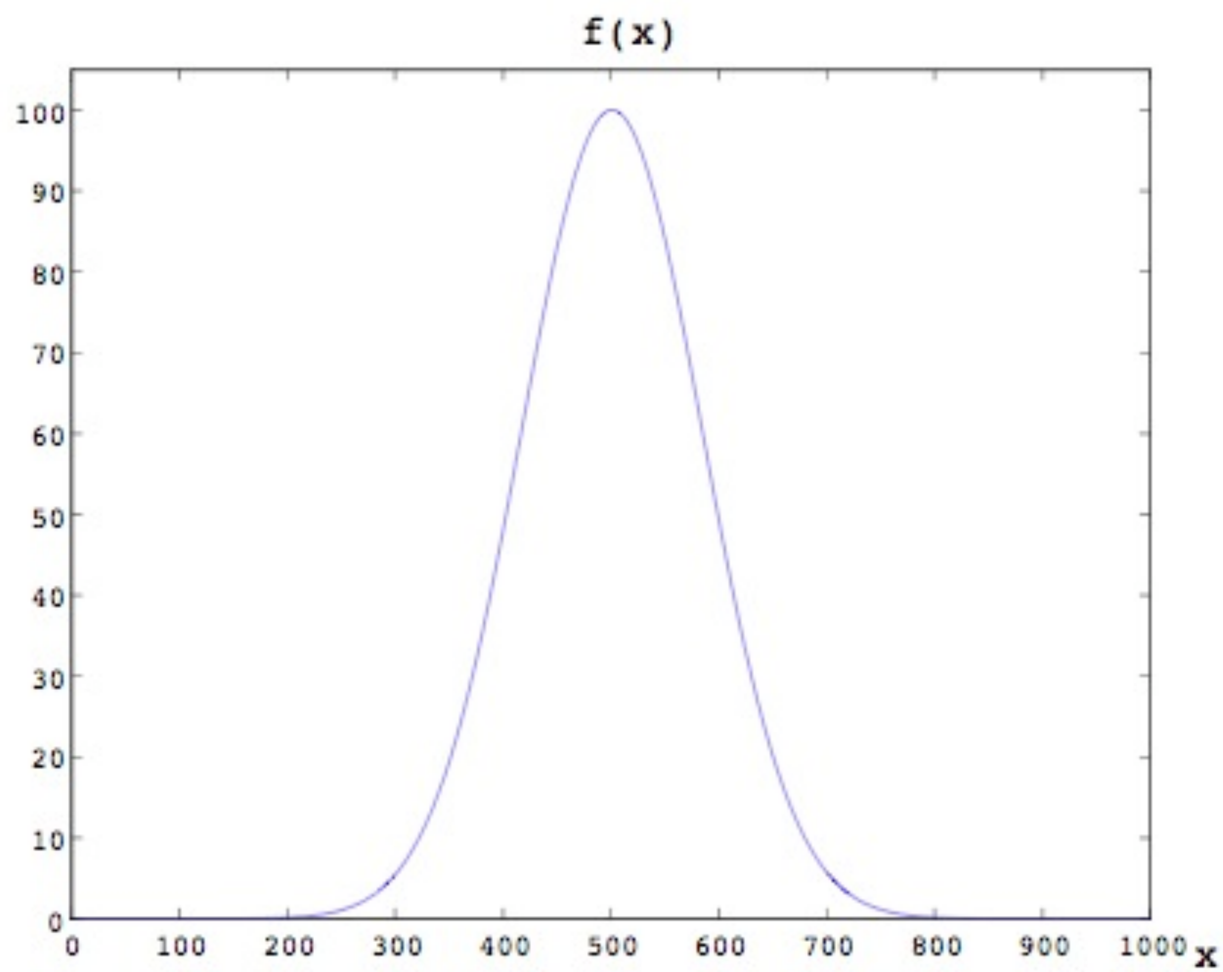

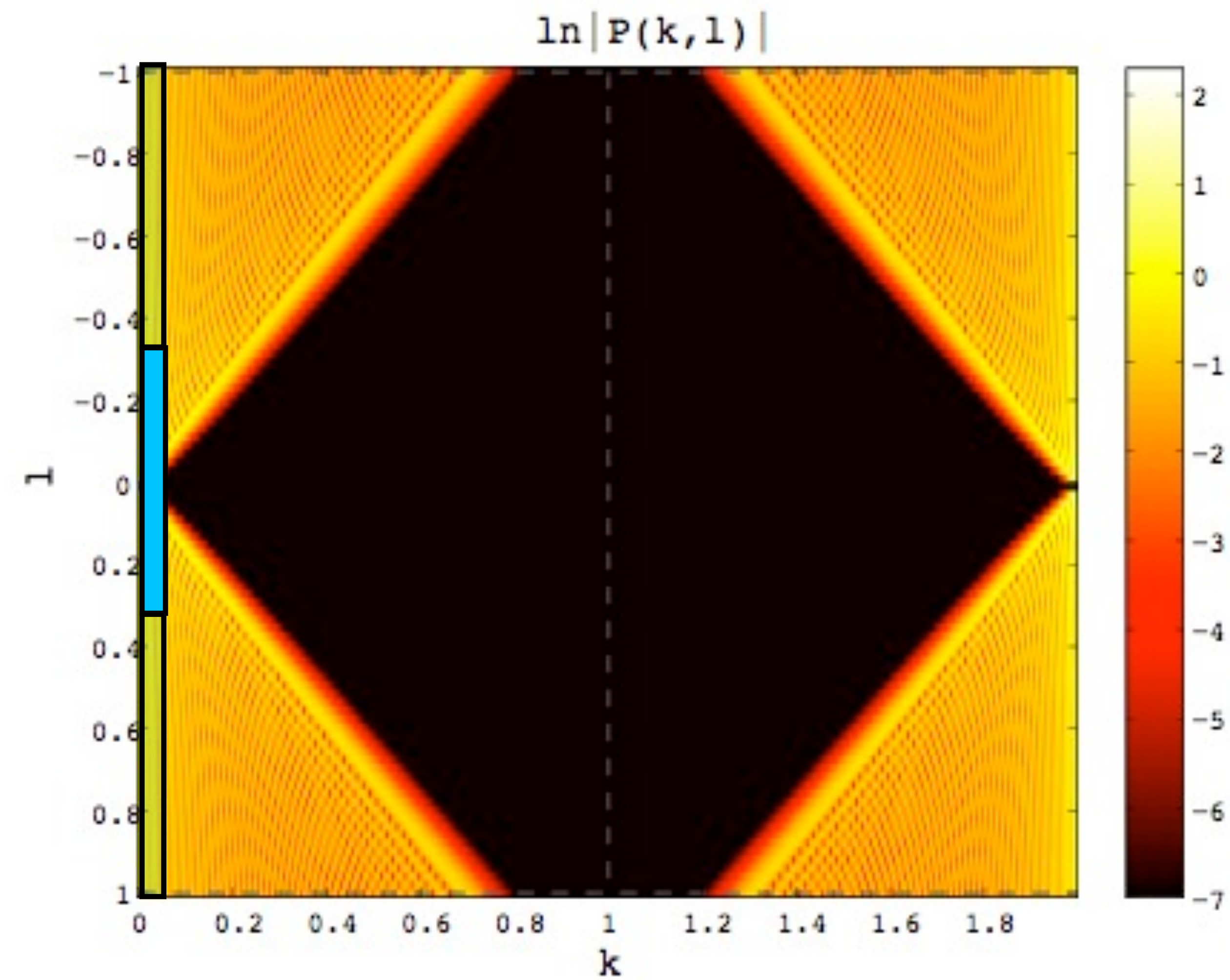

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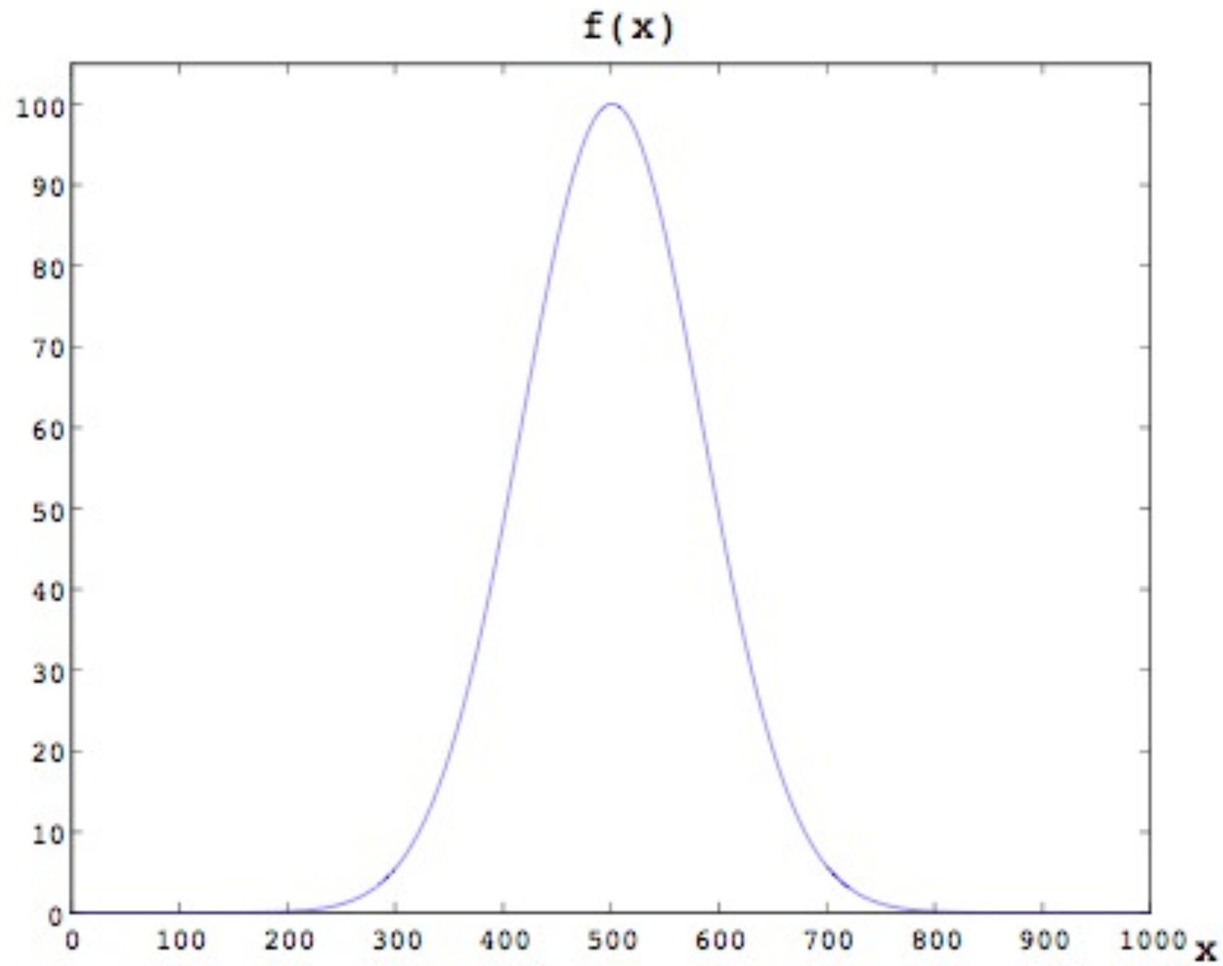

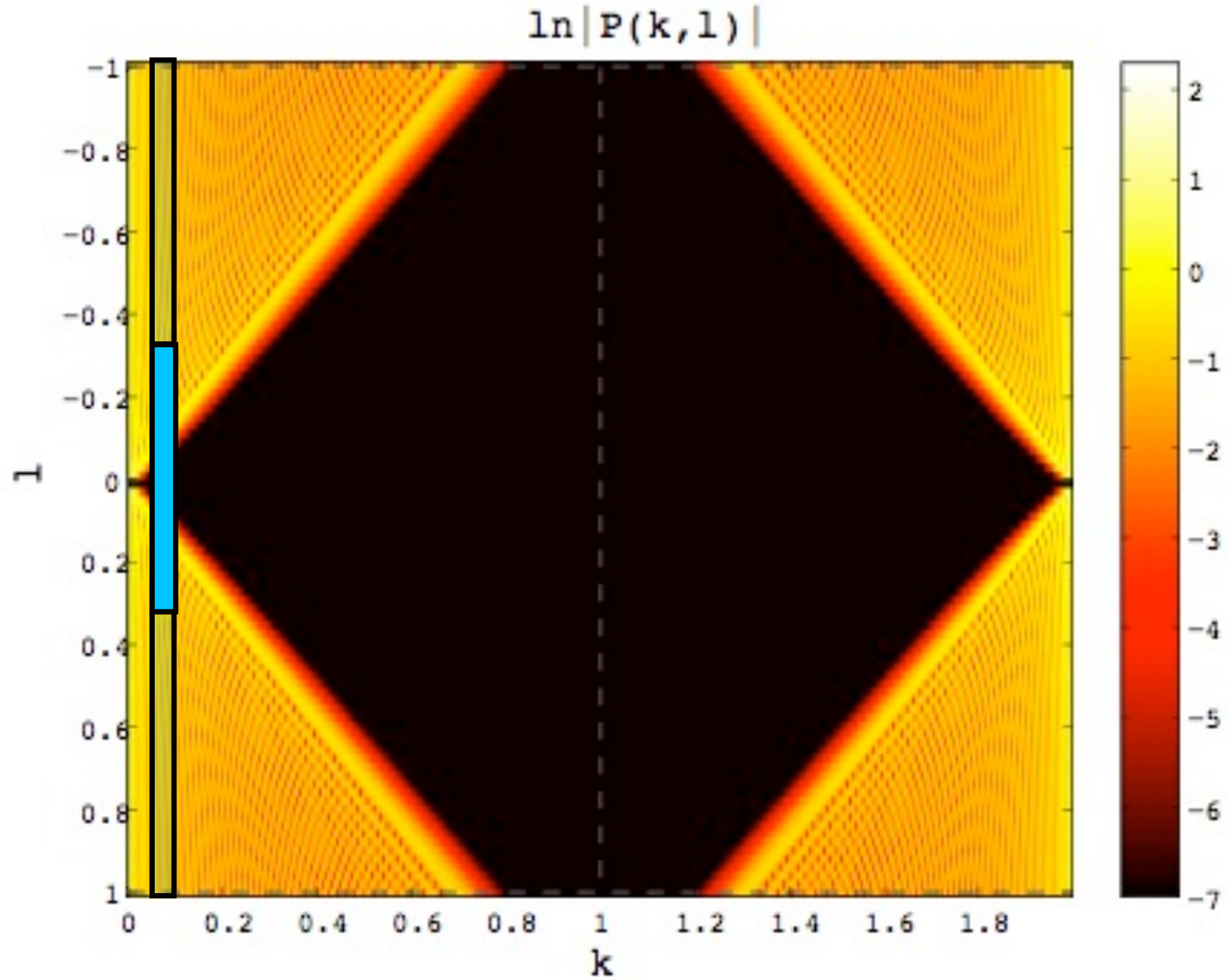

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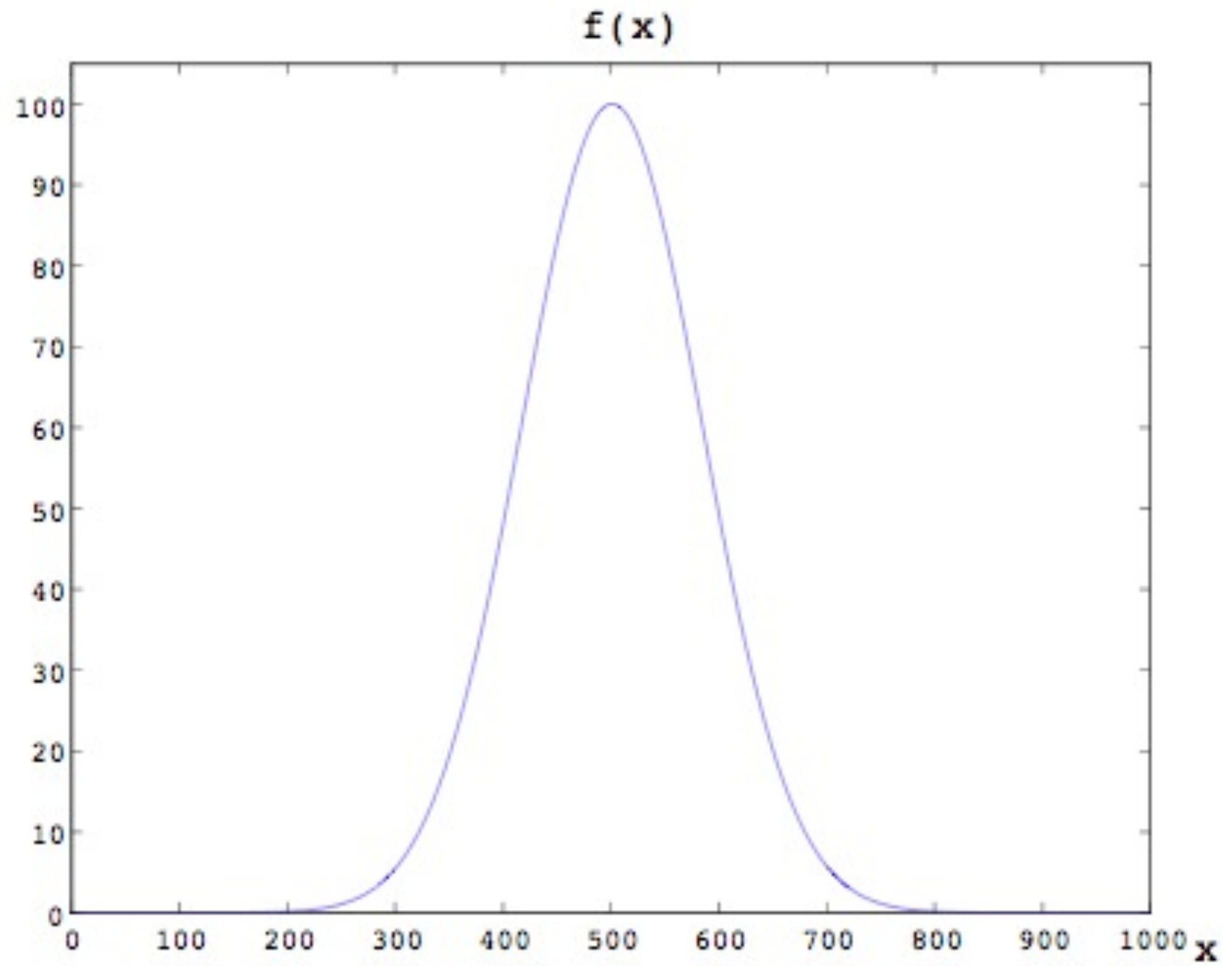

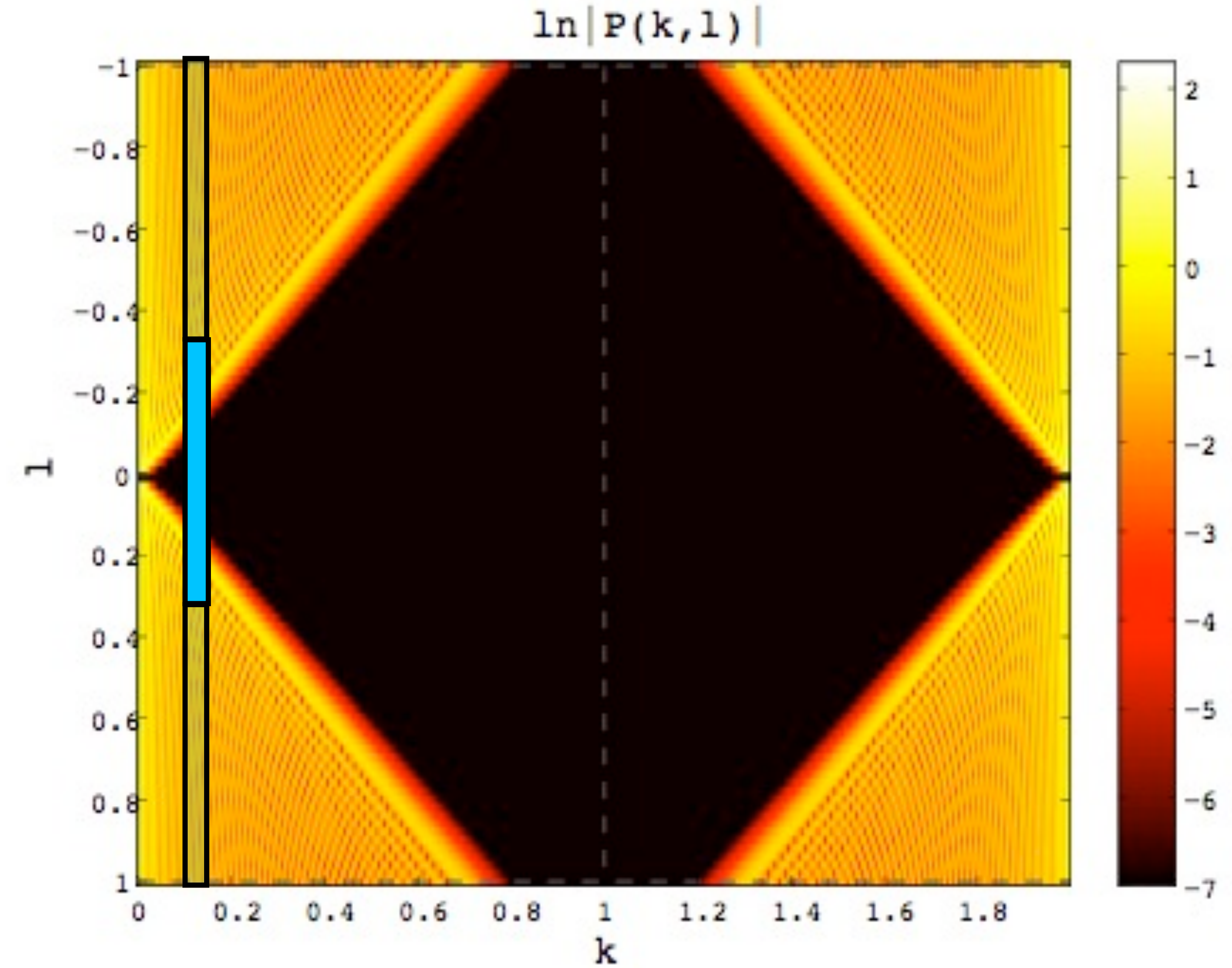

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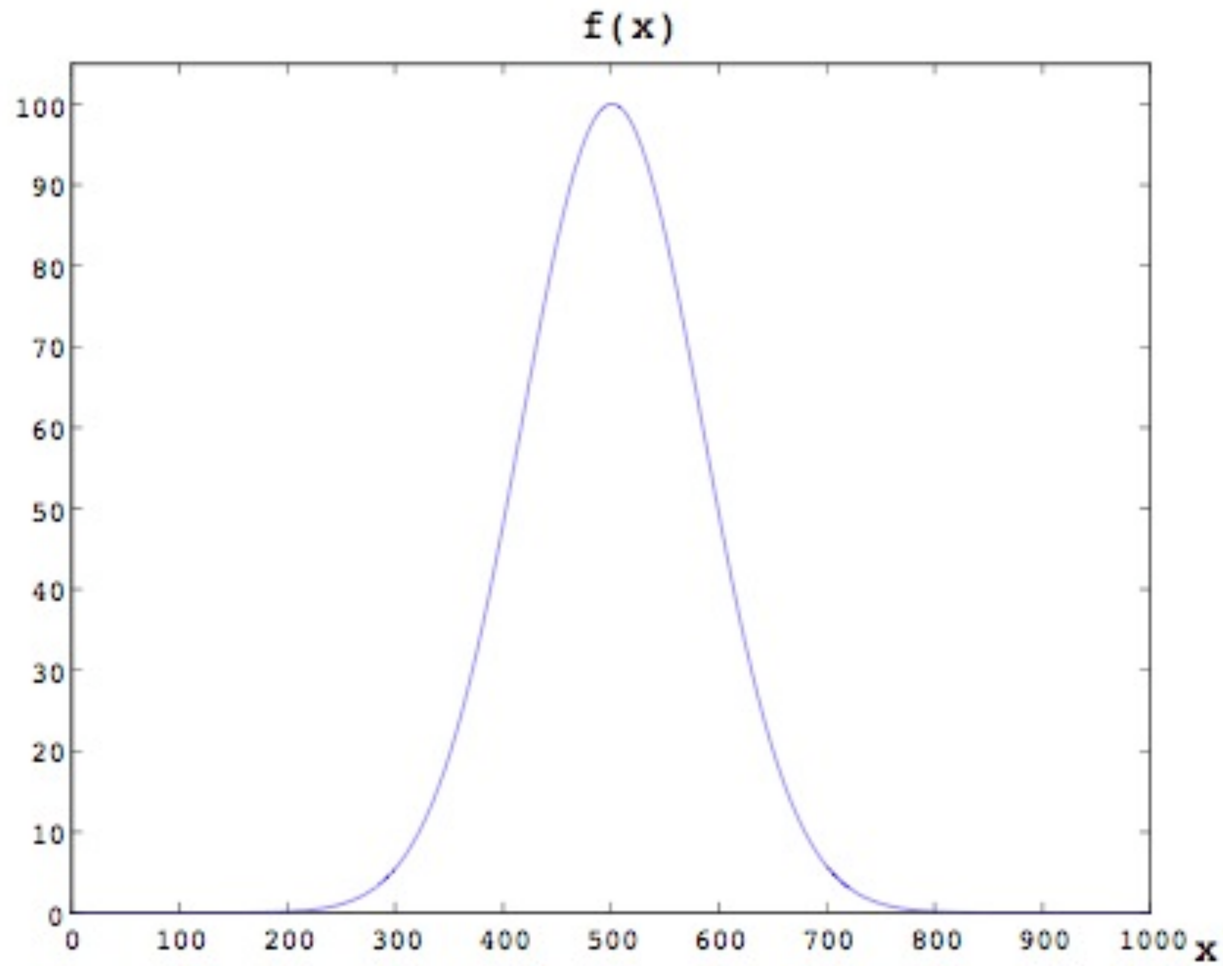

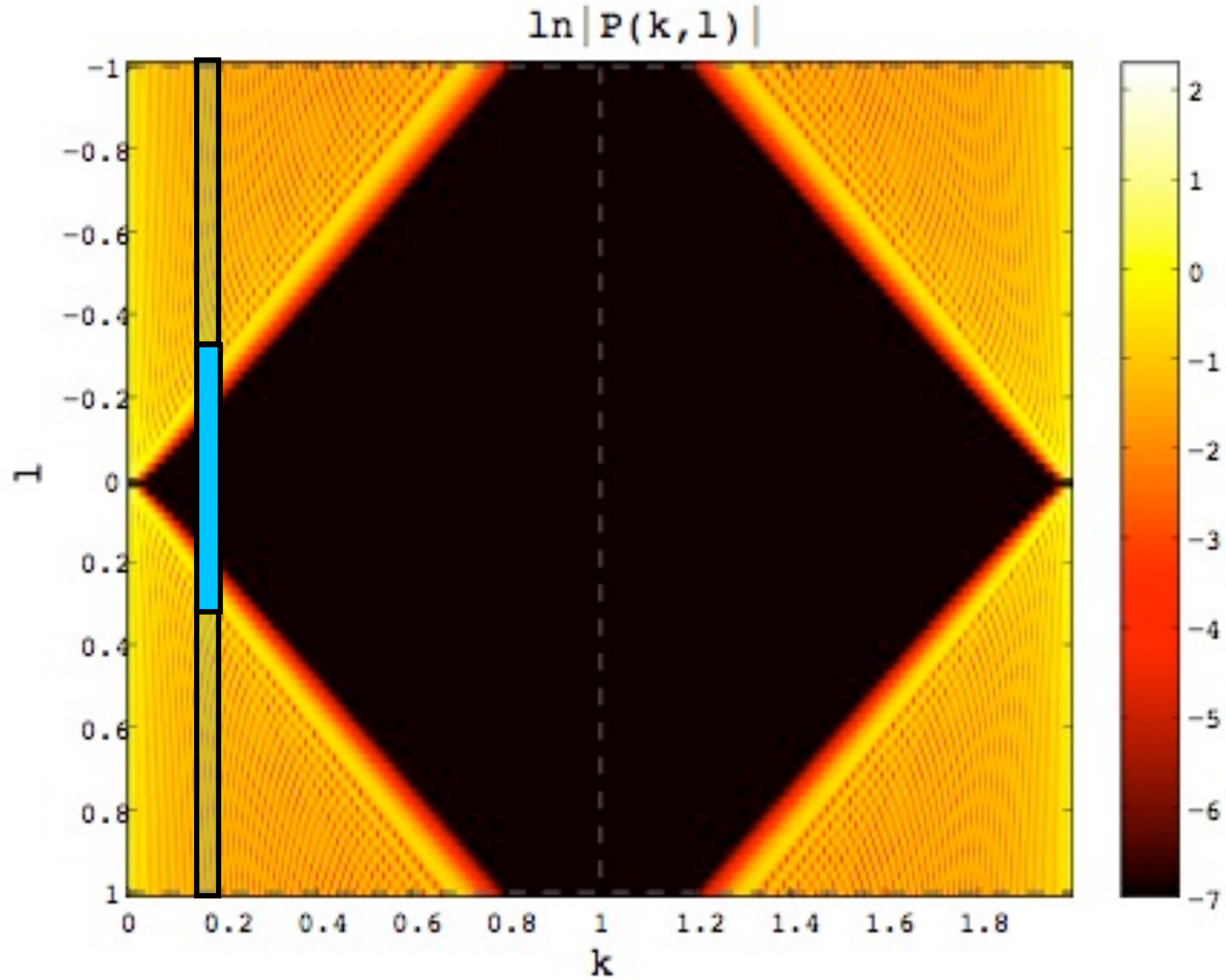

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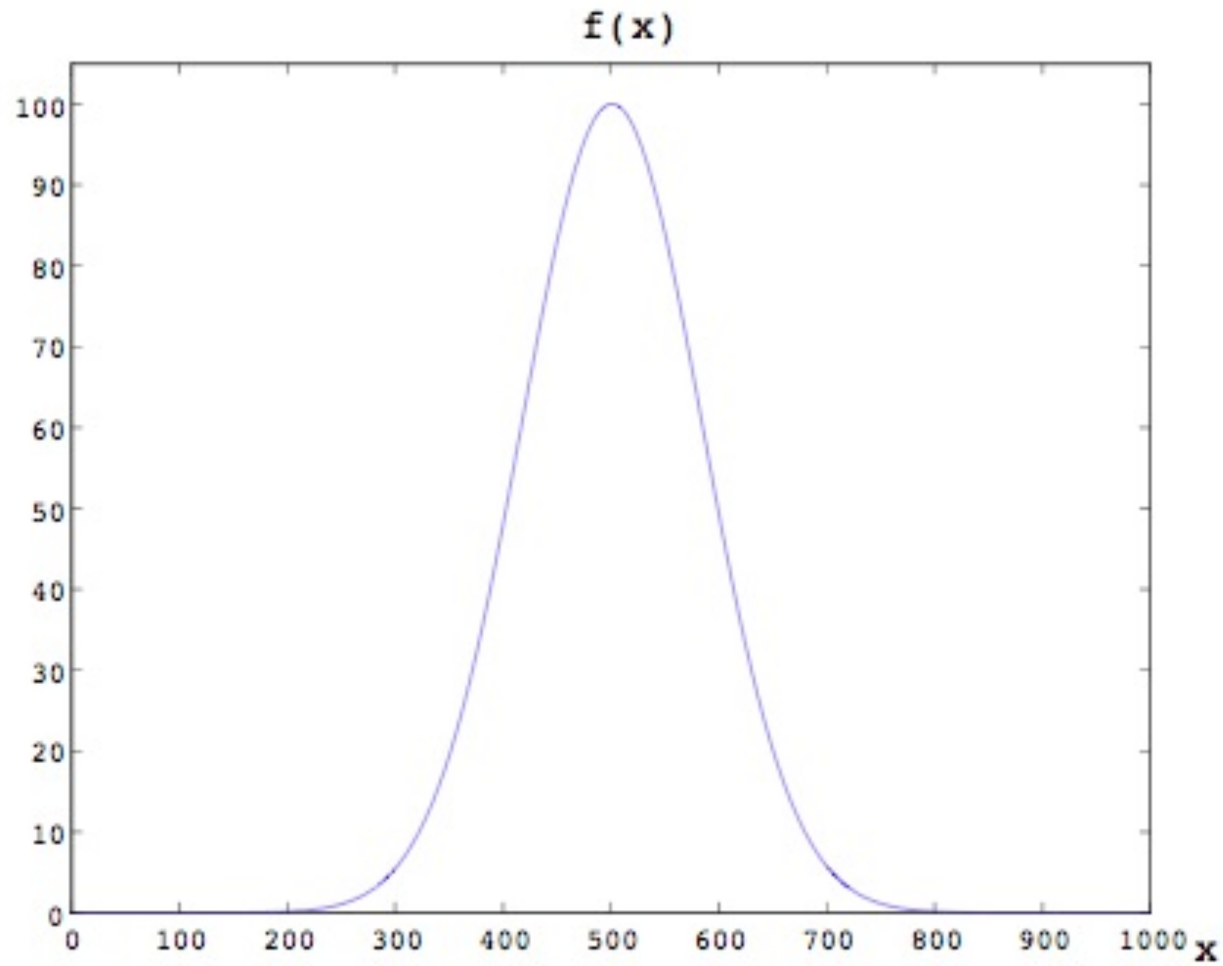

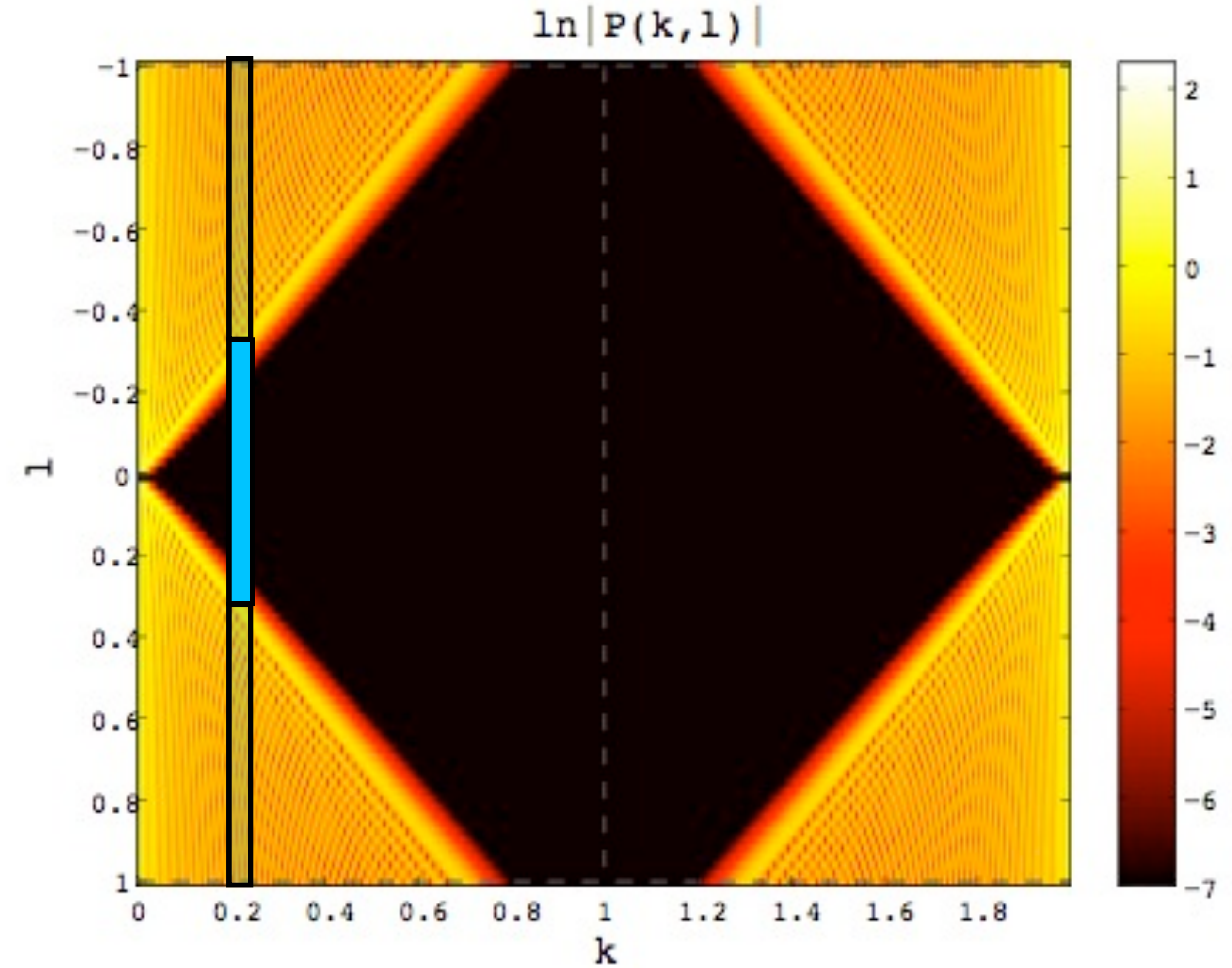

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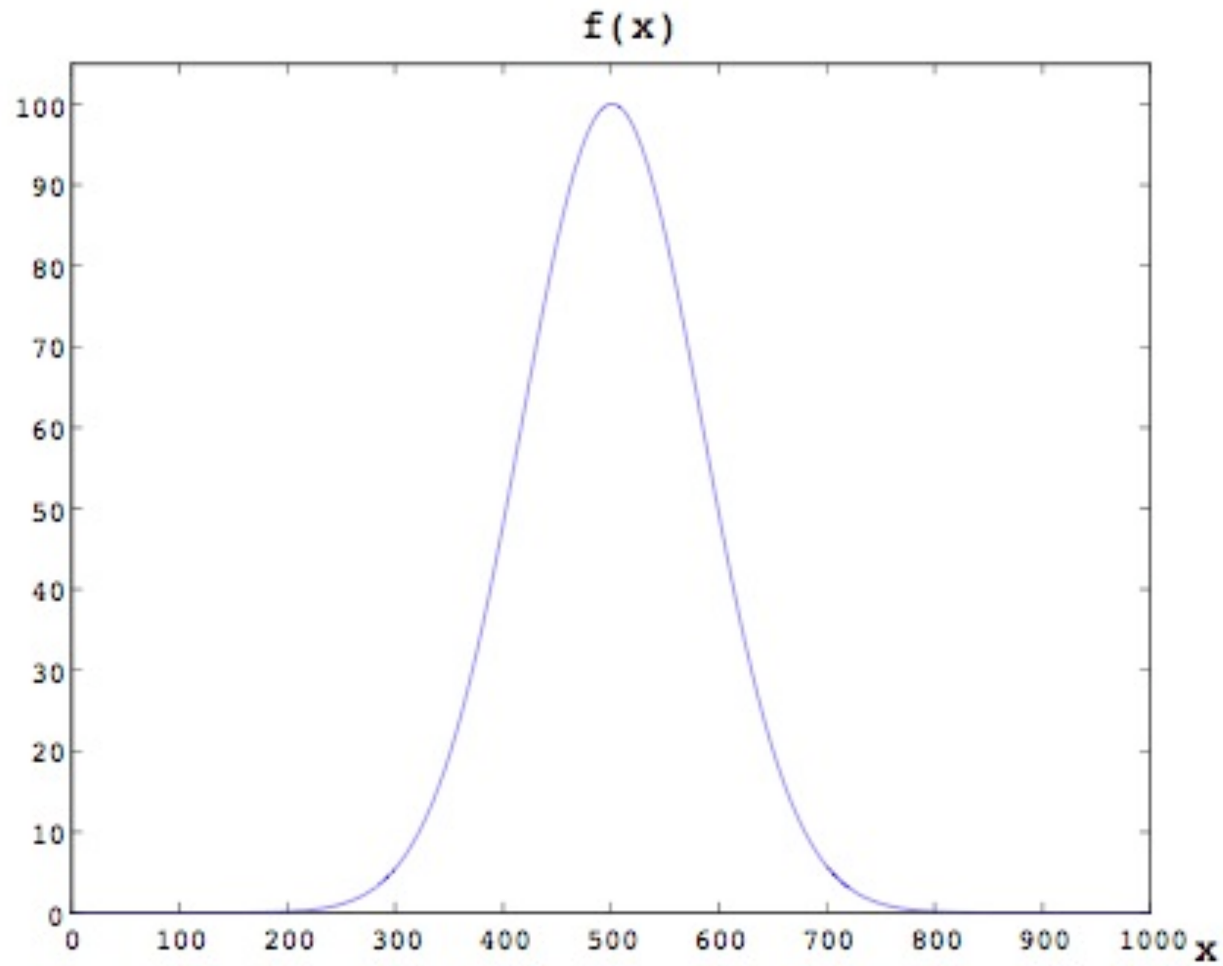

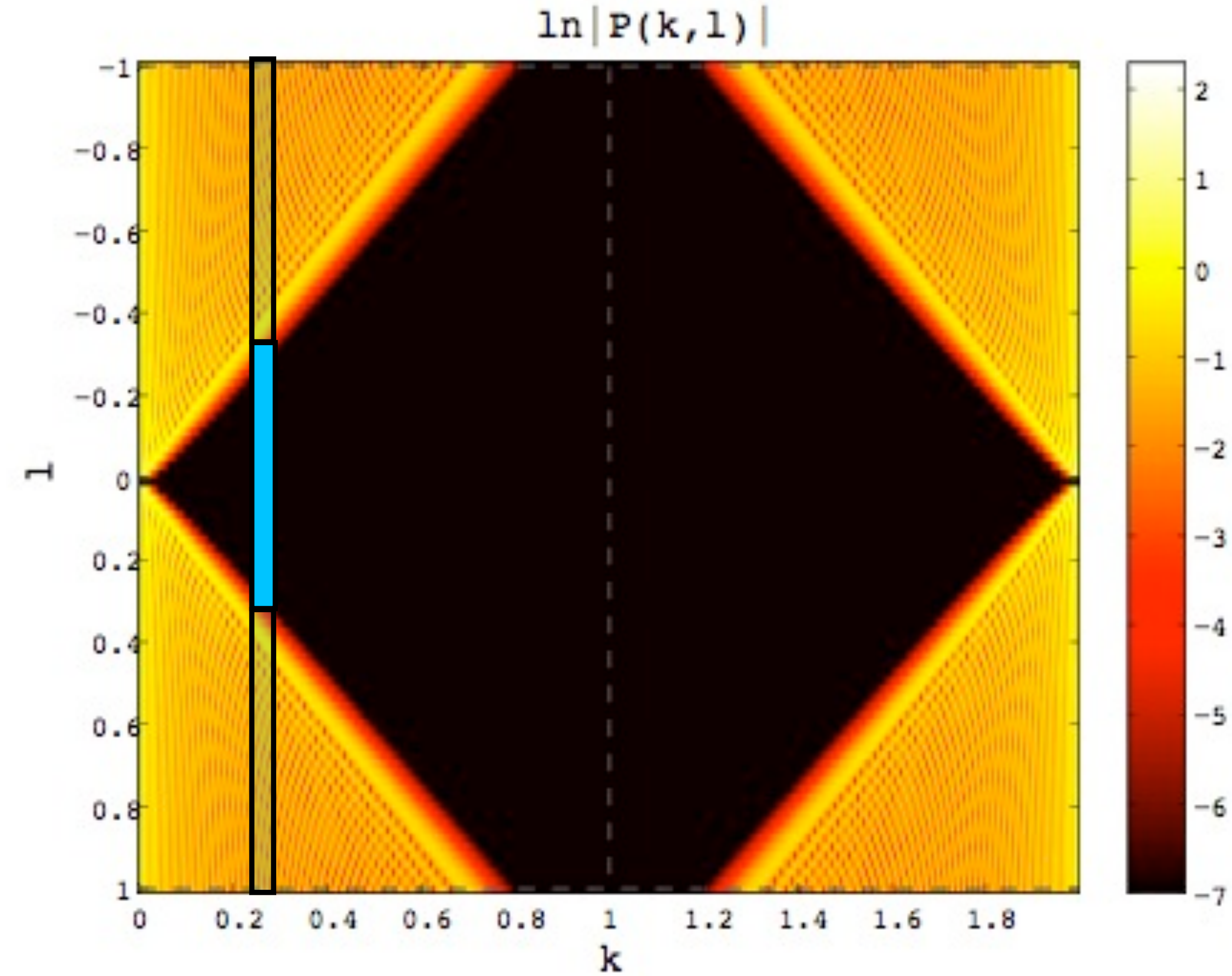

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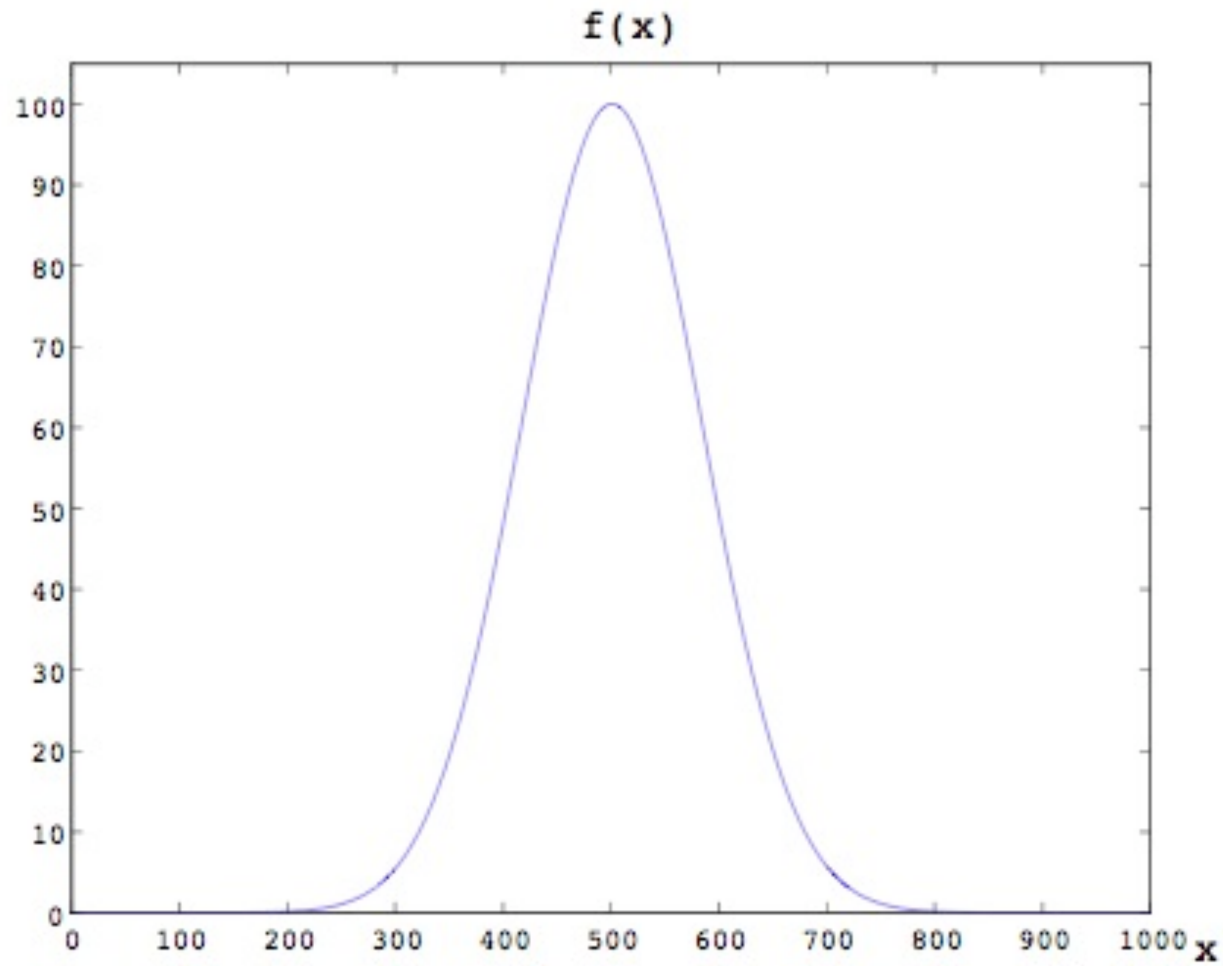

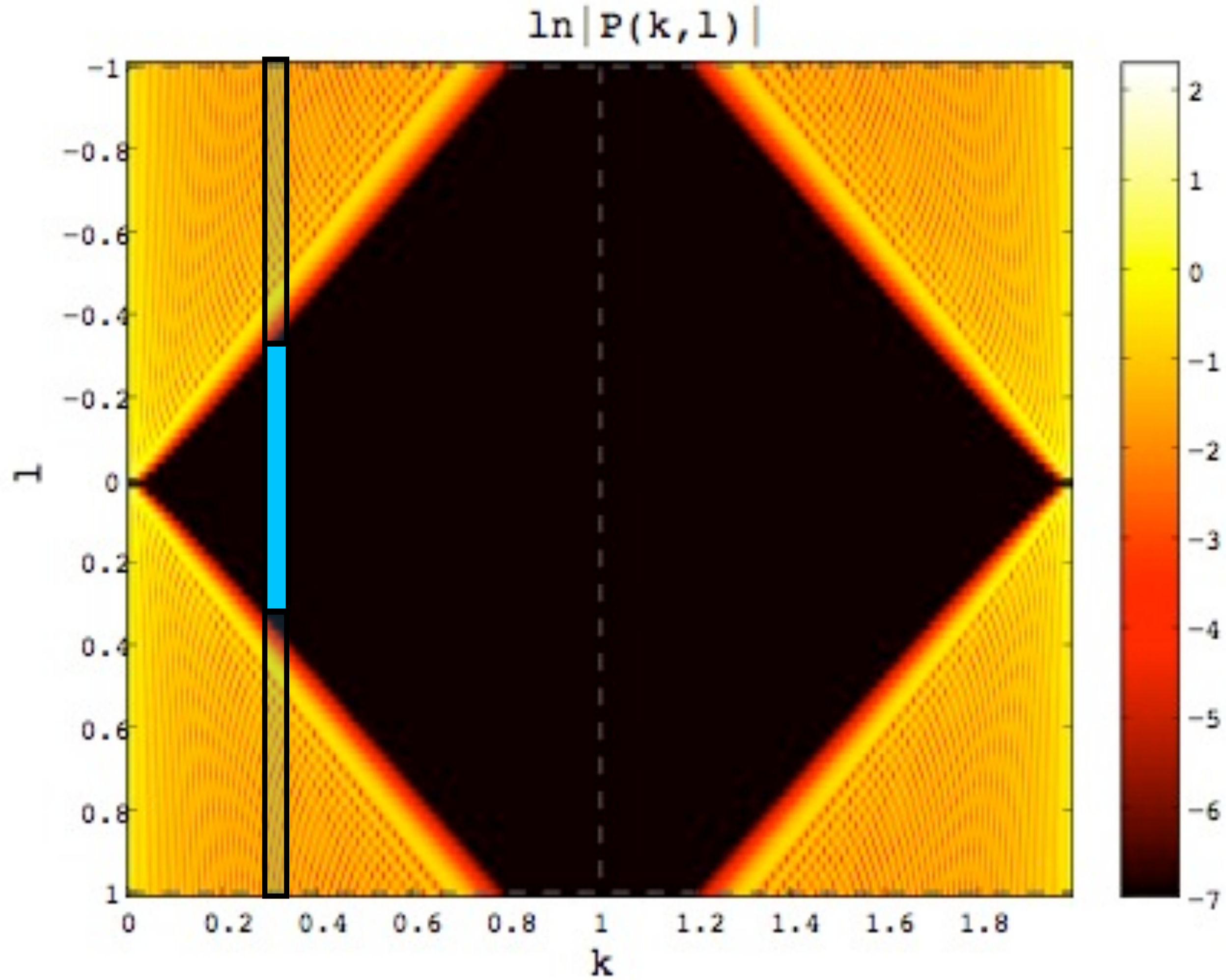

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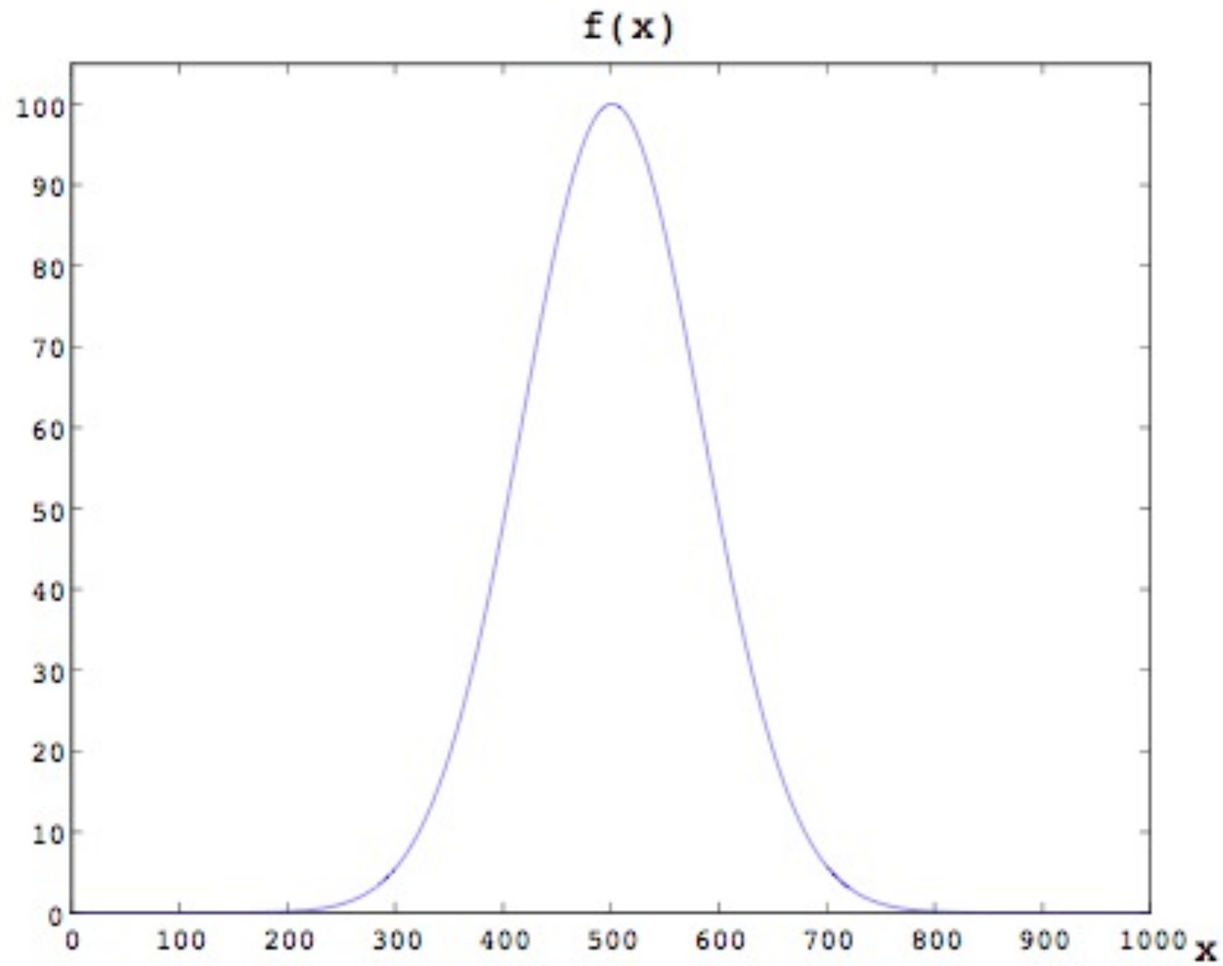

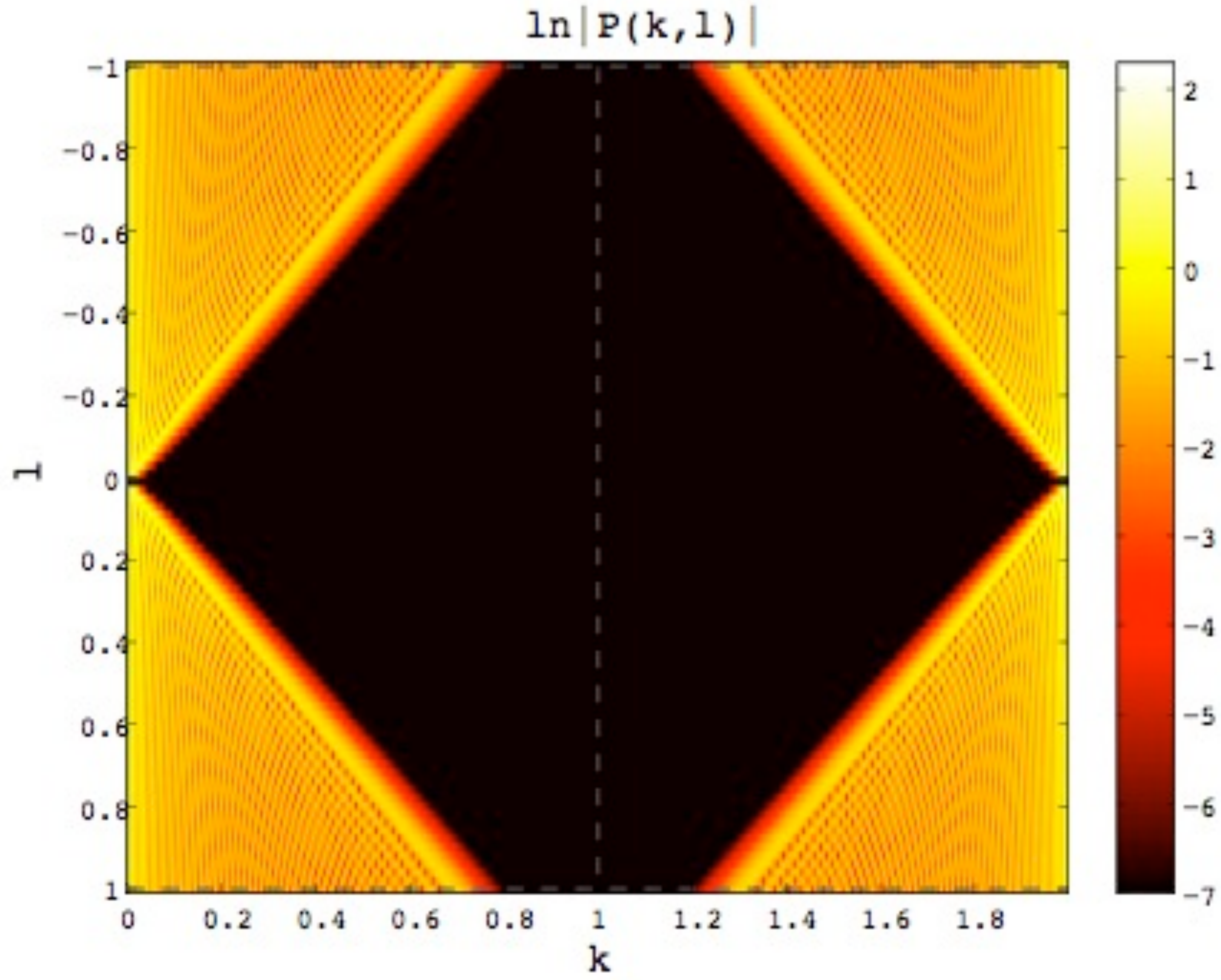

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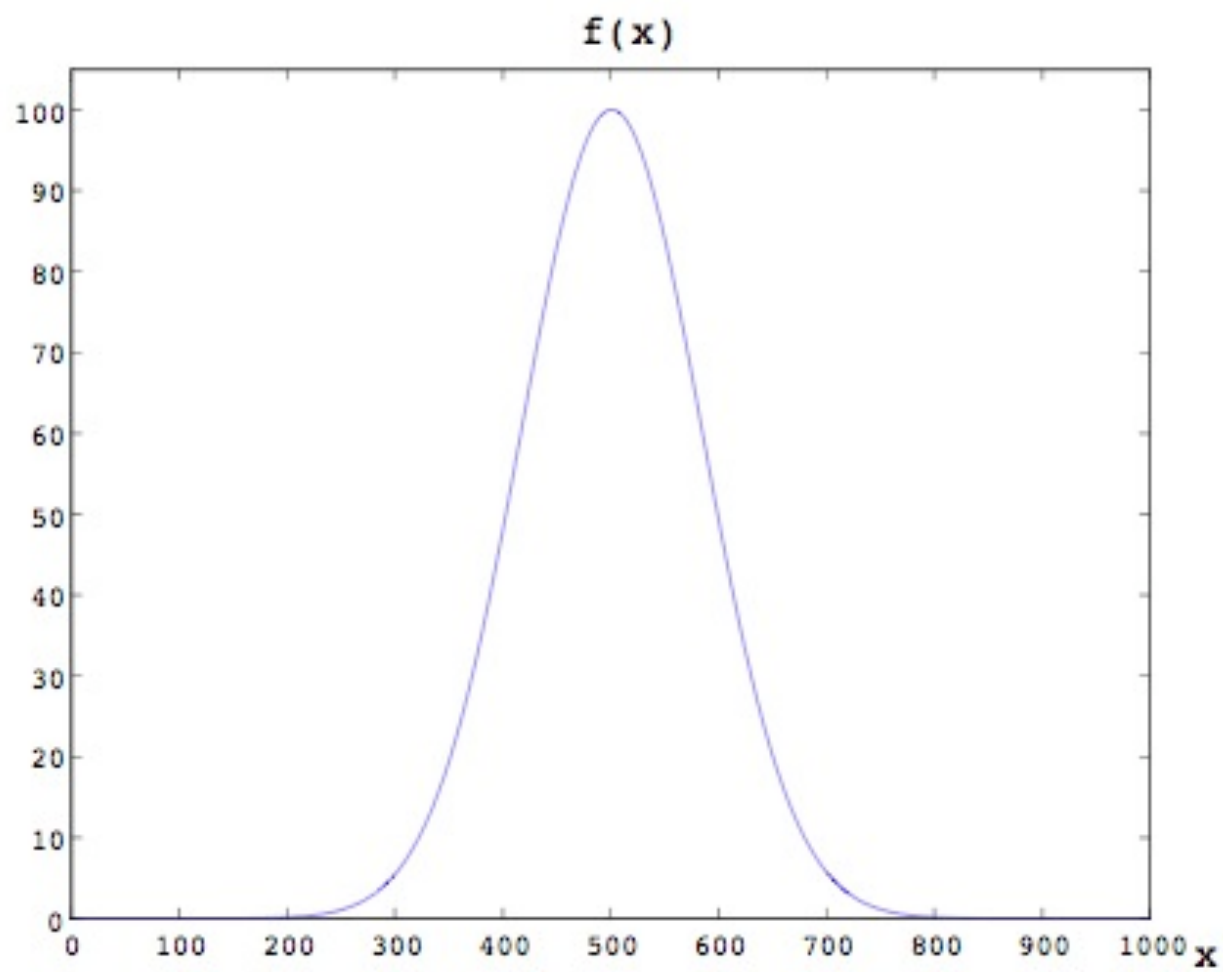

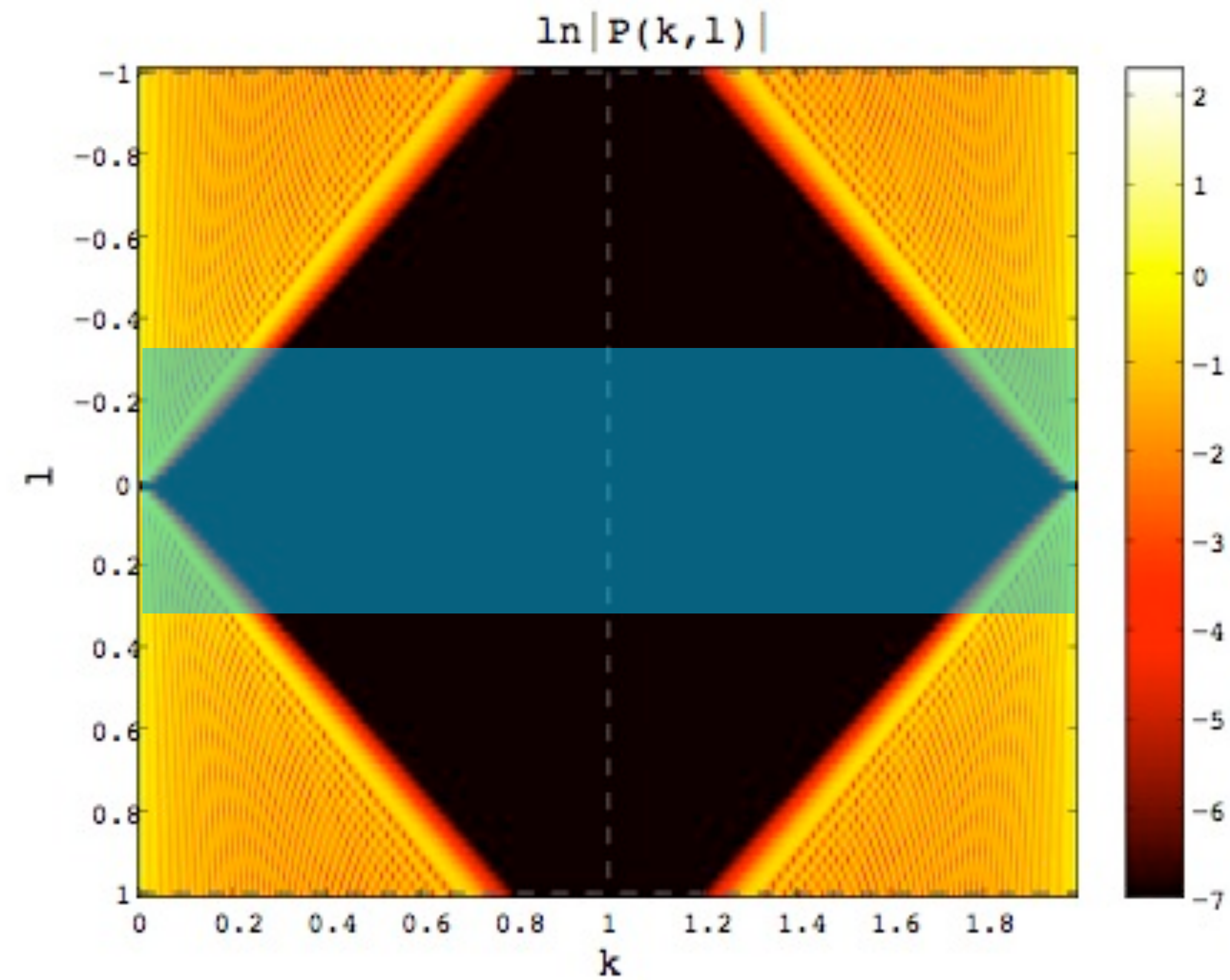

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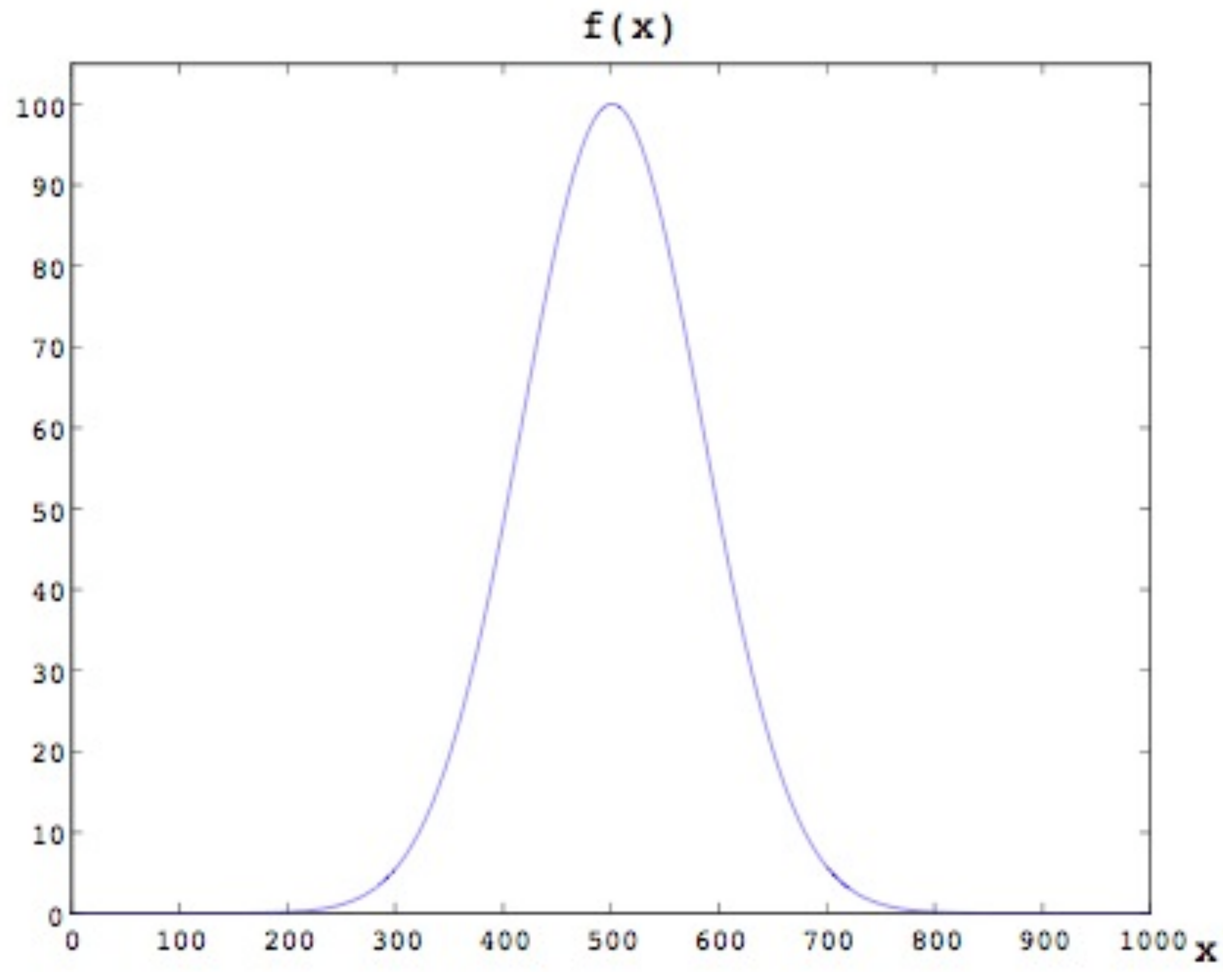

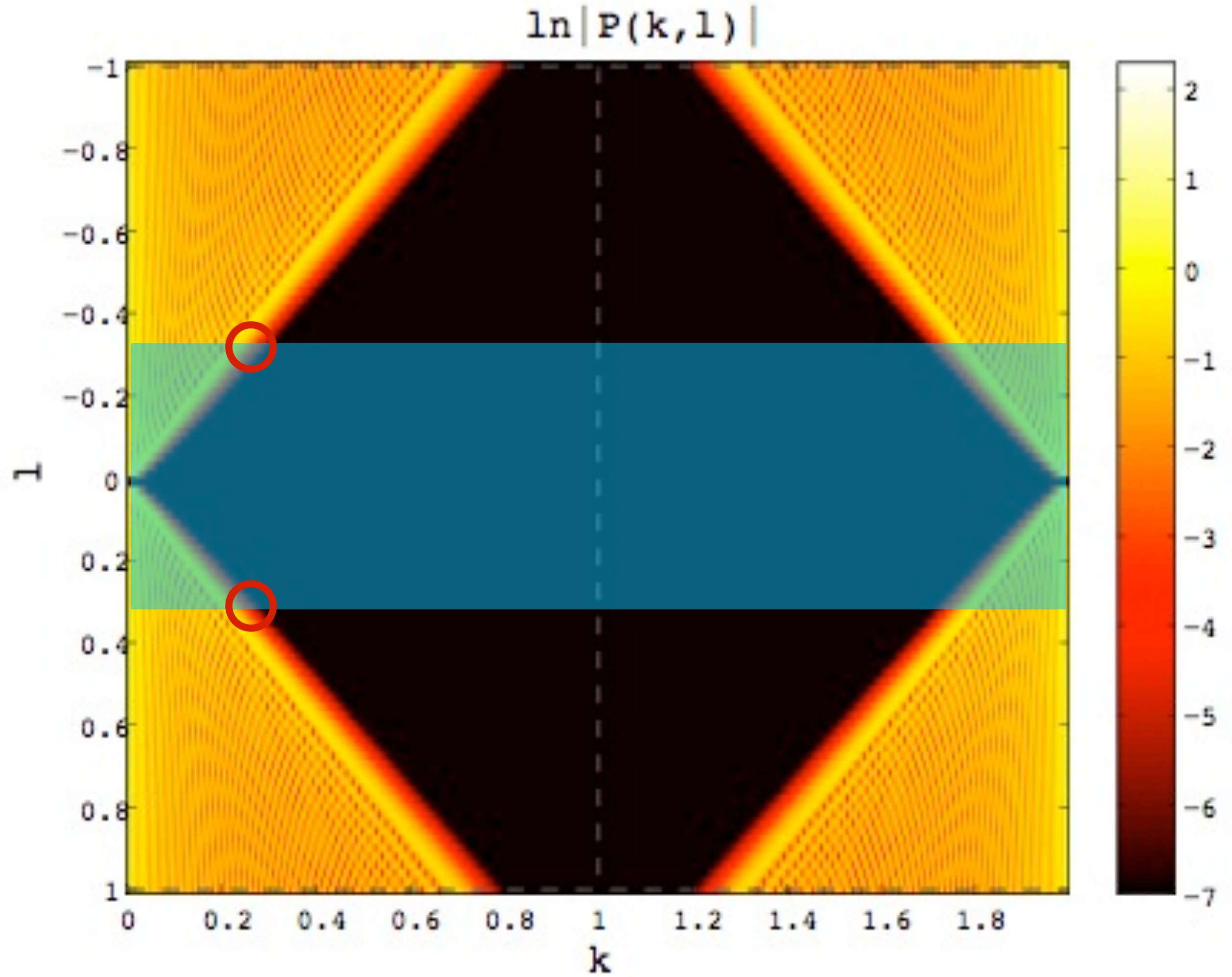

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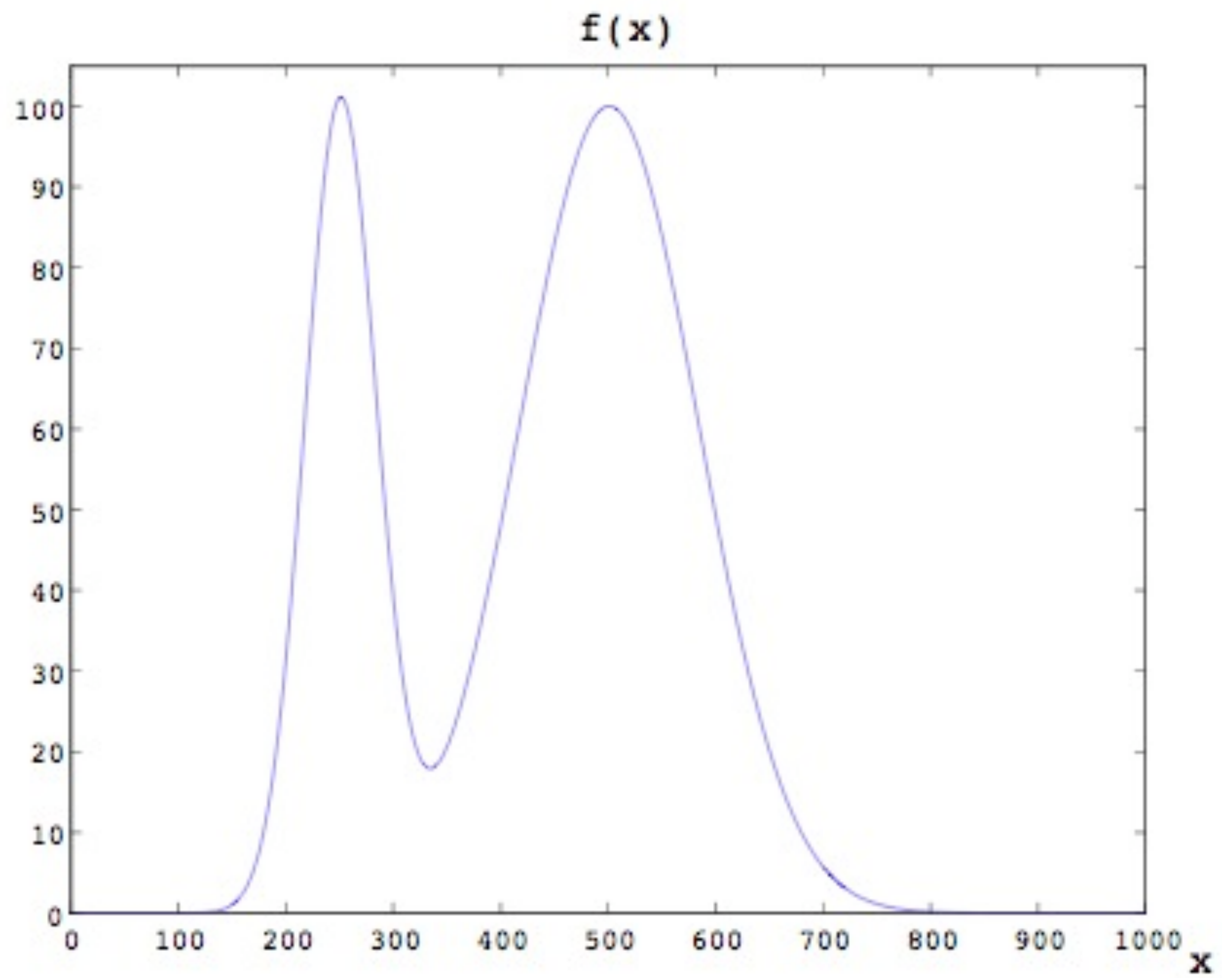

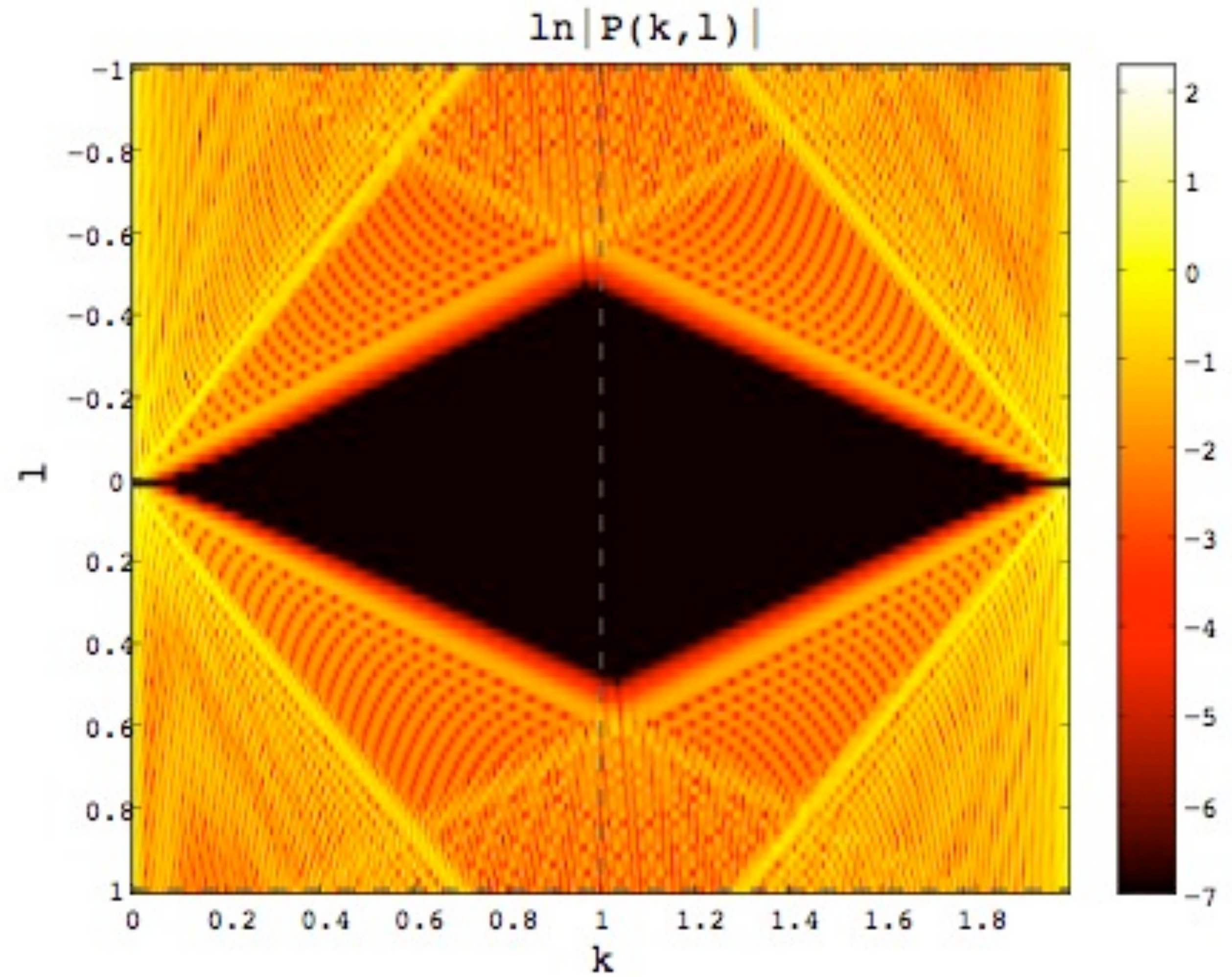

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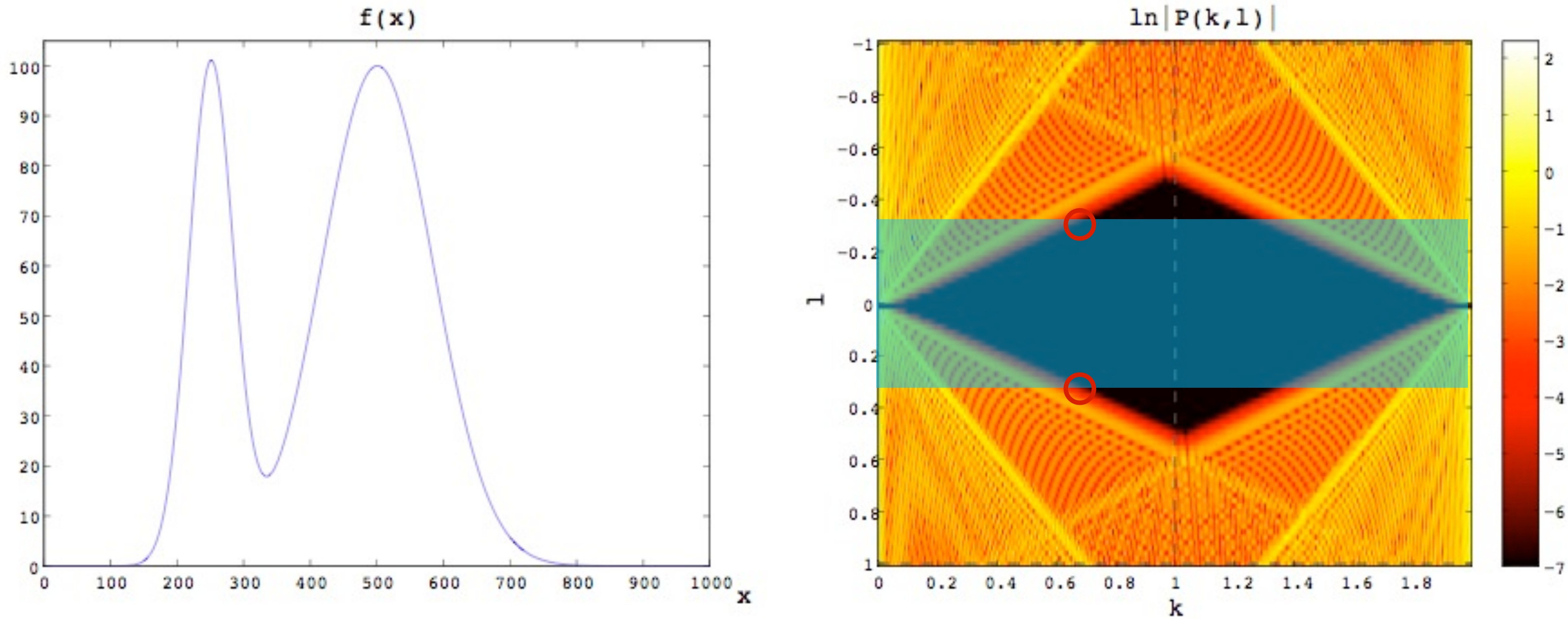

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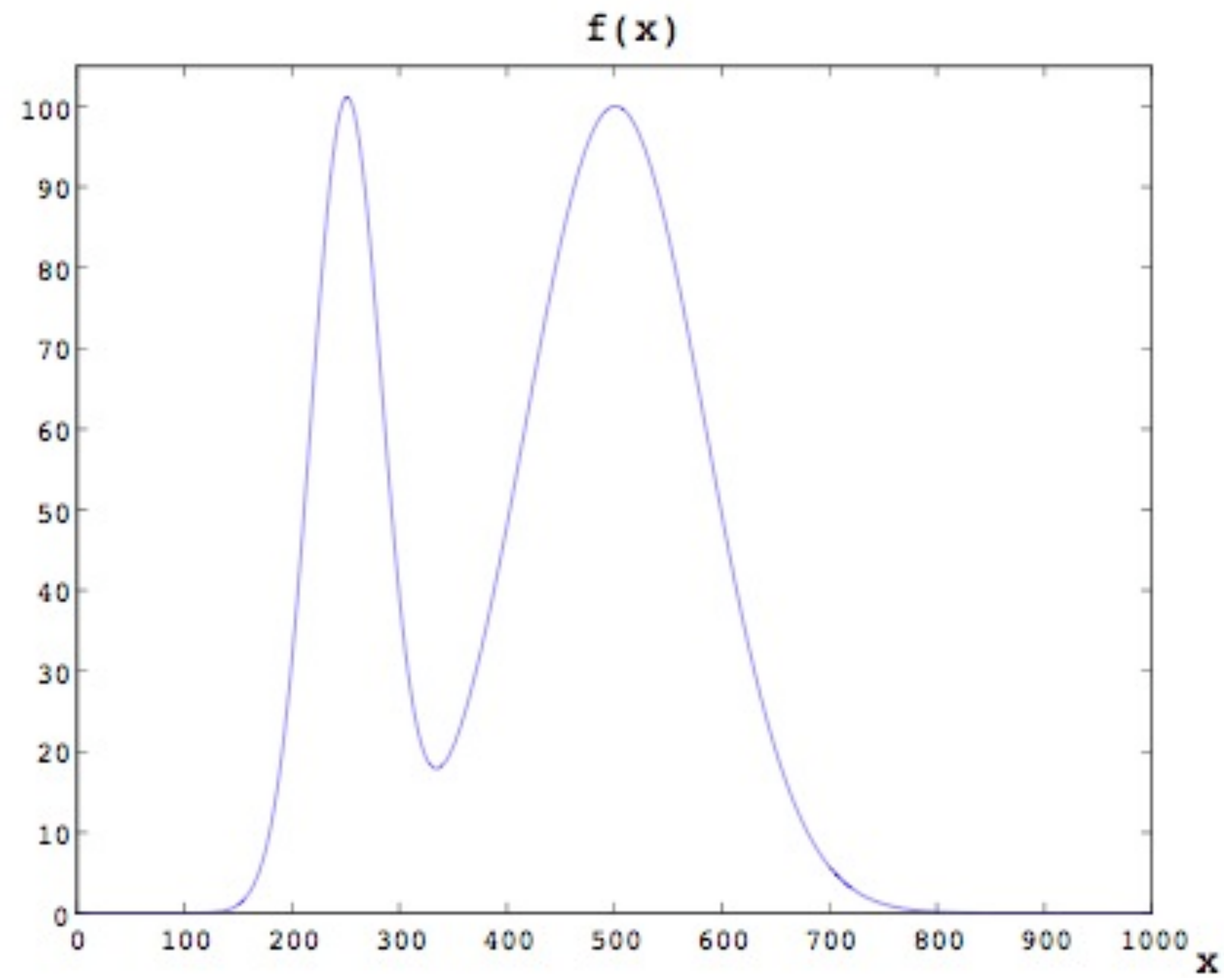
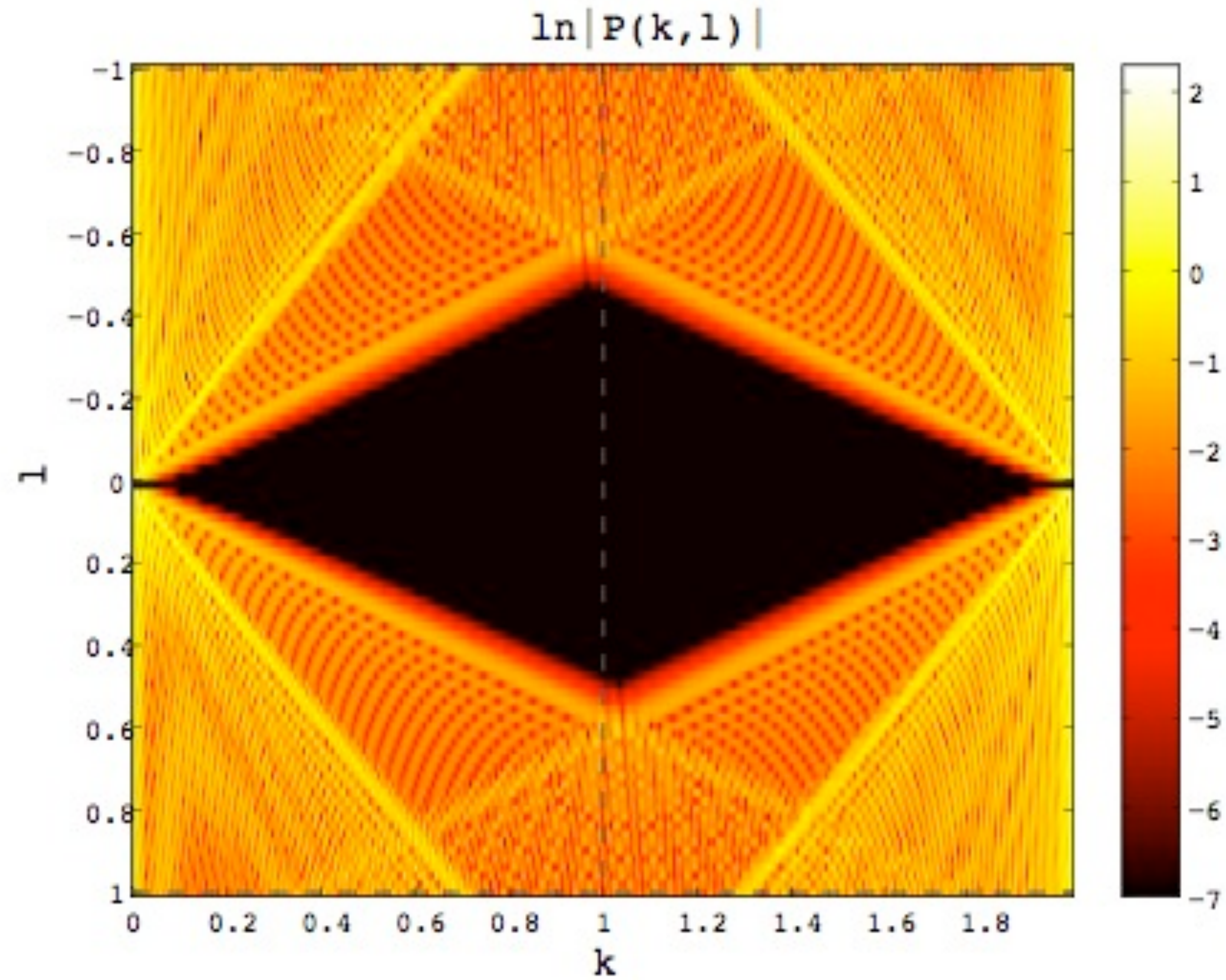
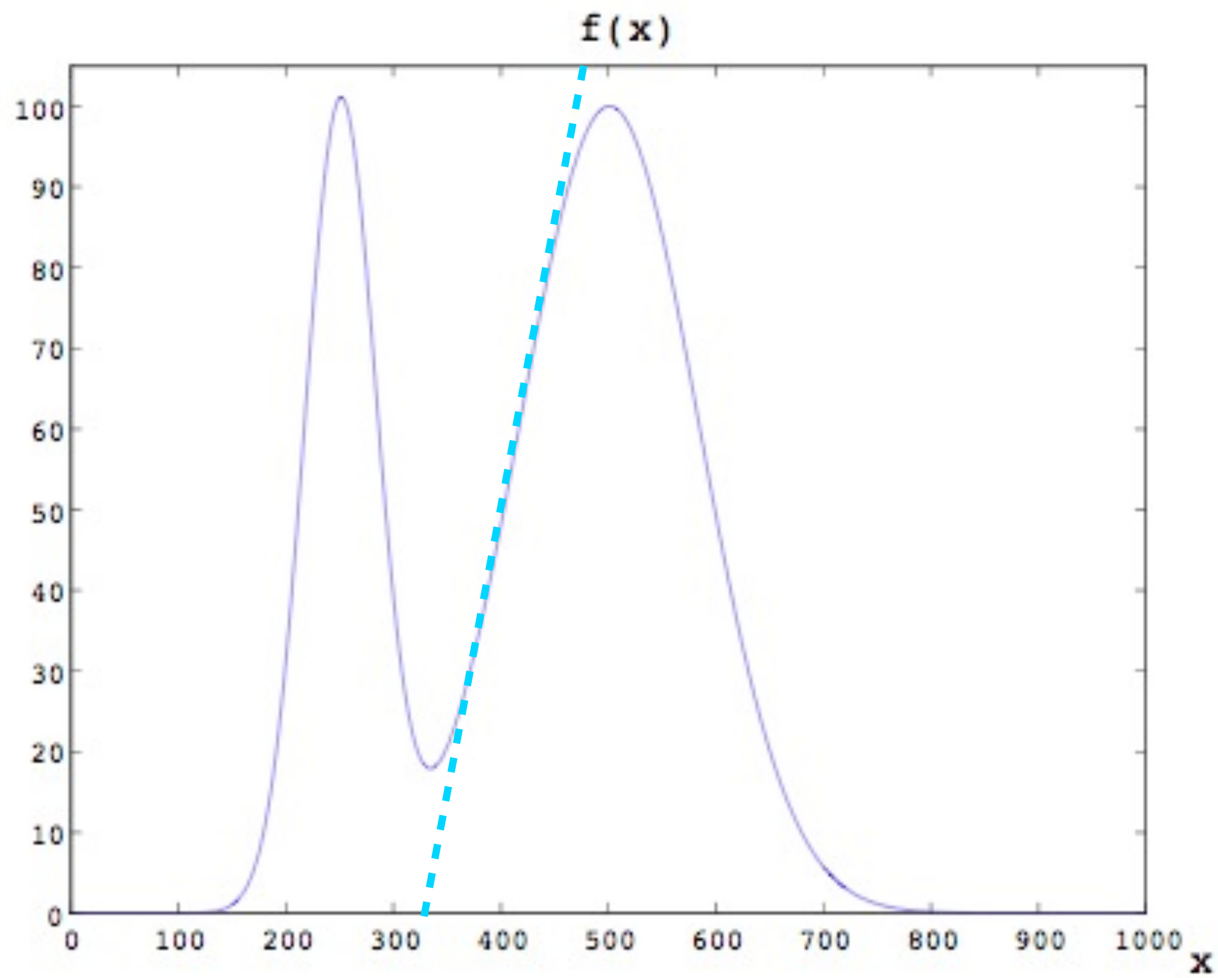
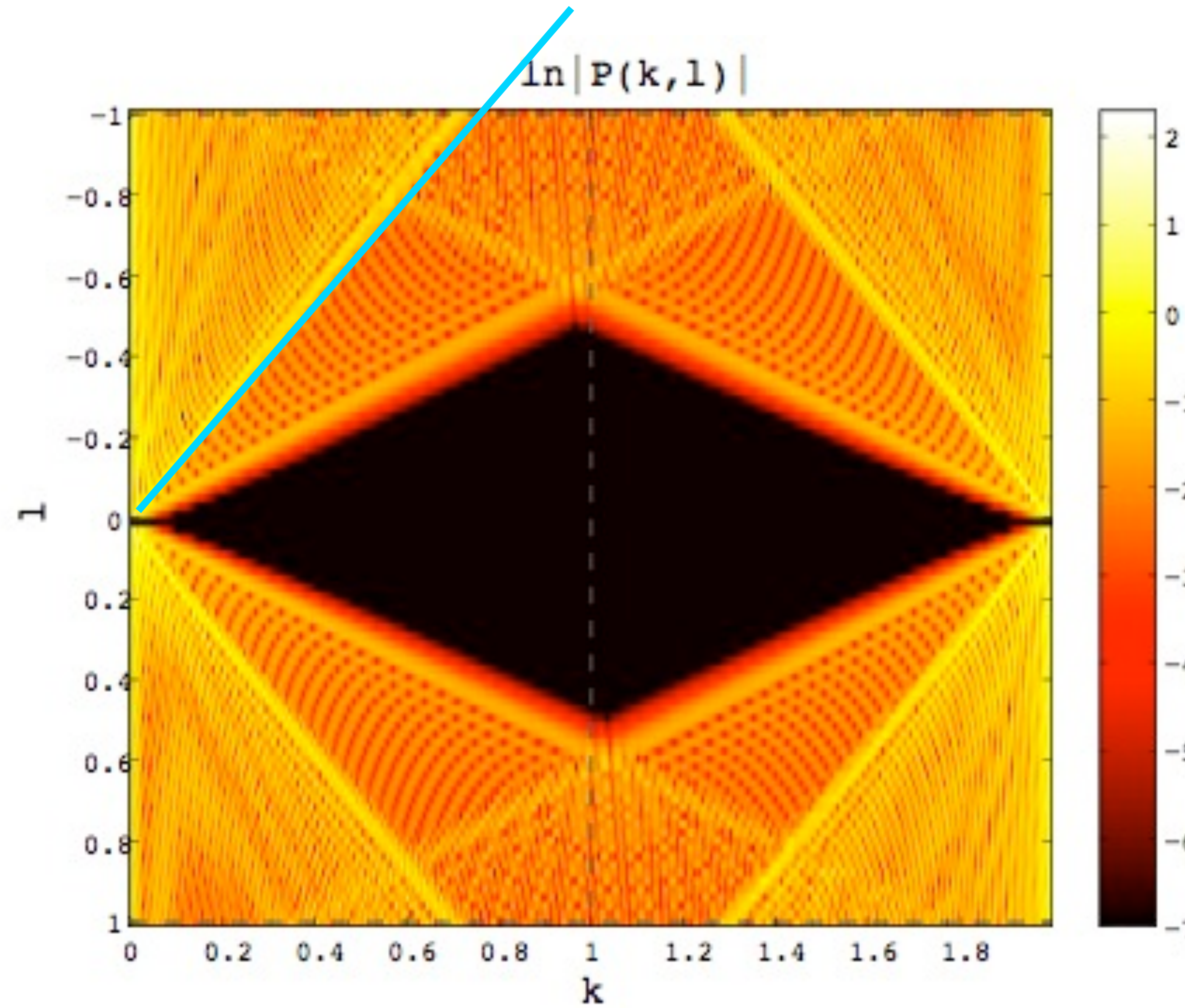




Visualizing P(k,l)

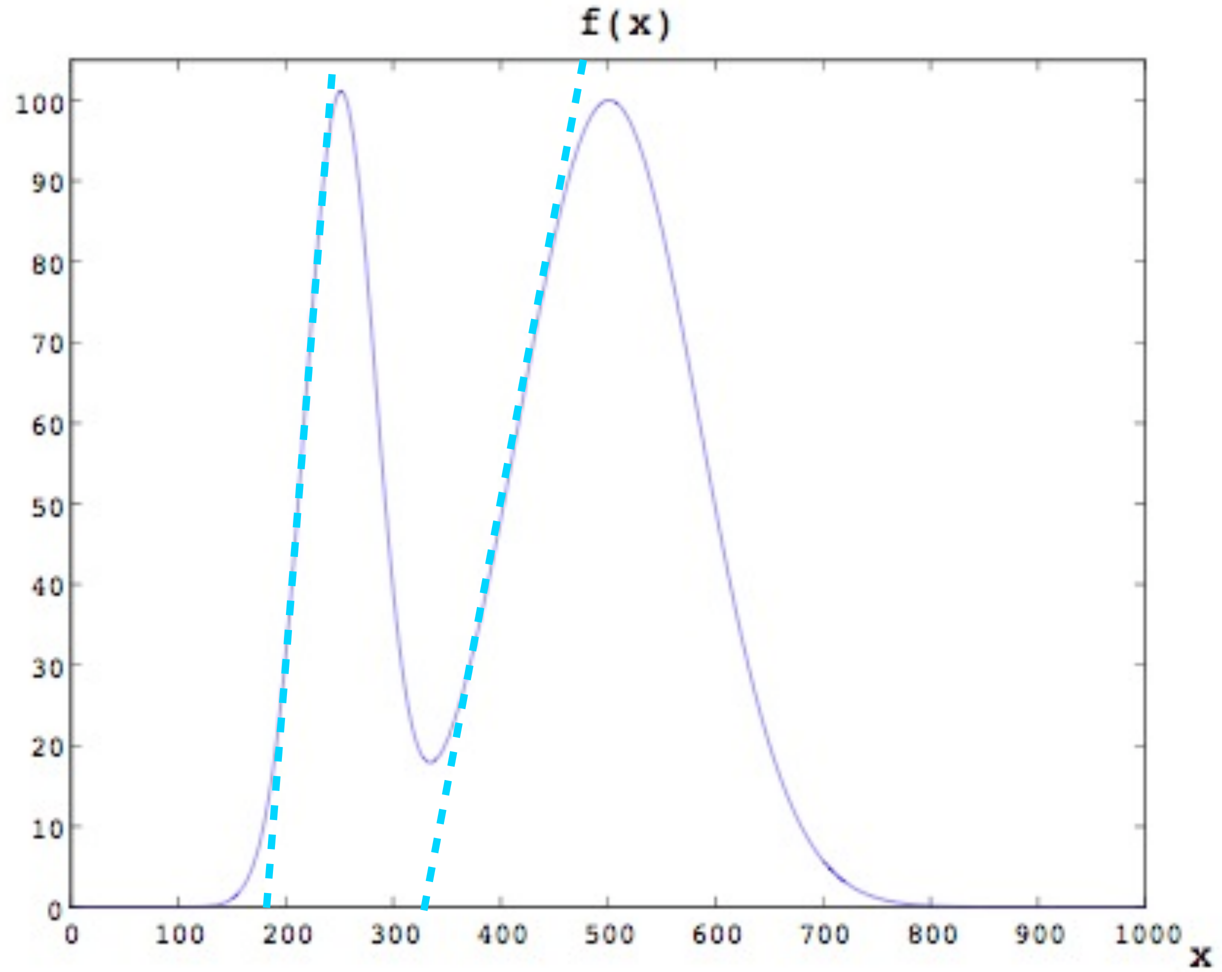
- Slopes of lines in P(k,l) are related to 1/f'(x)

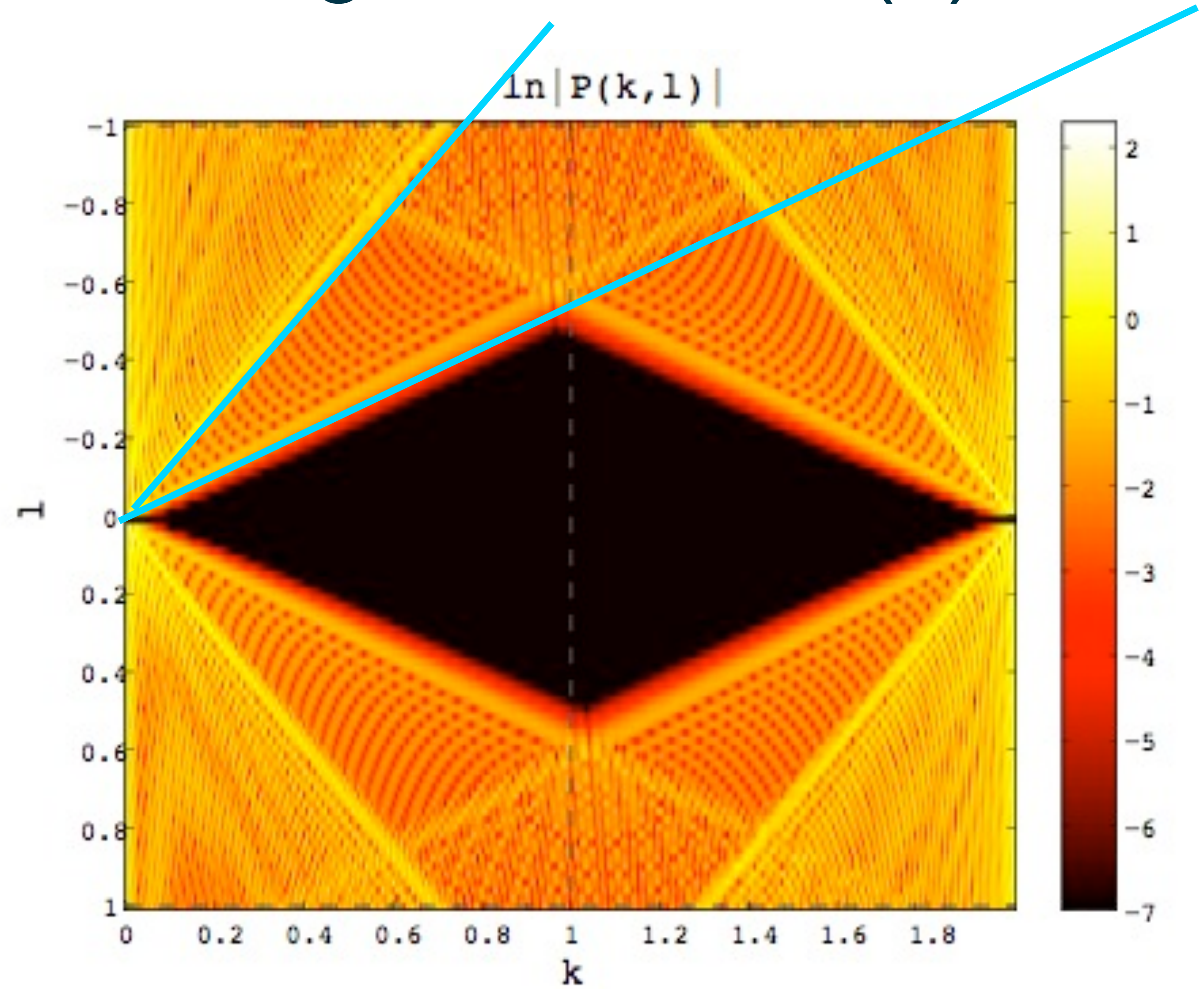




Visualizing P(k,l)

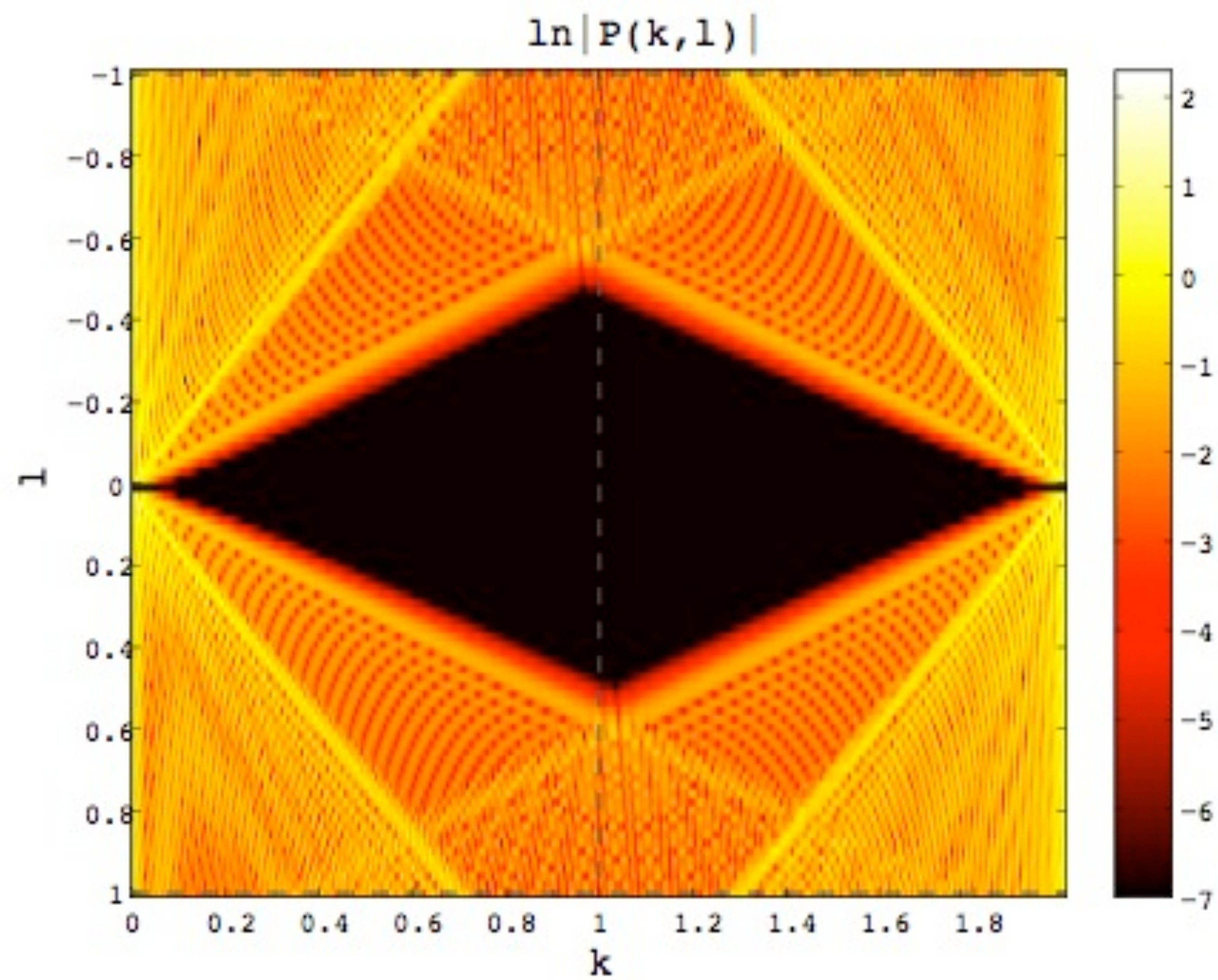
- Slopes of lines in P(k,l) are related to 1/f'(x)
- Extremal slopes bounding the wedge are 1/max(f')





Method of stationary phase

$$P(k,l) = \int_R e^{i(l \cdot f(x) - k \cdot x)} dx$$



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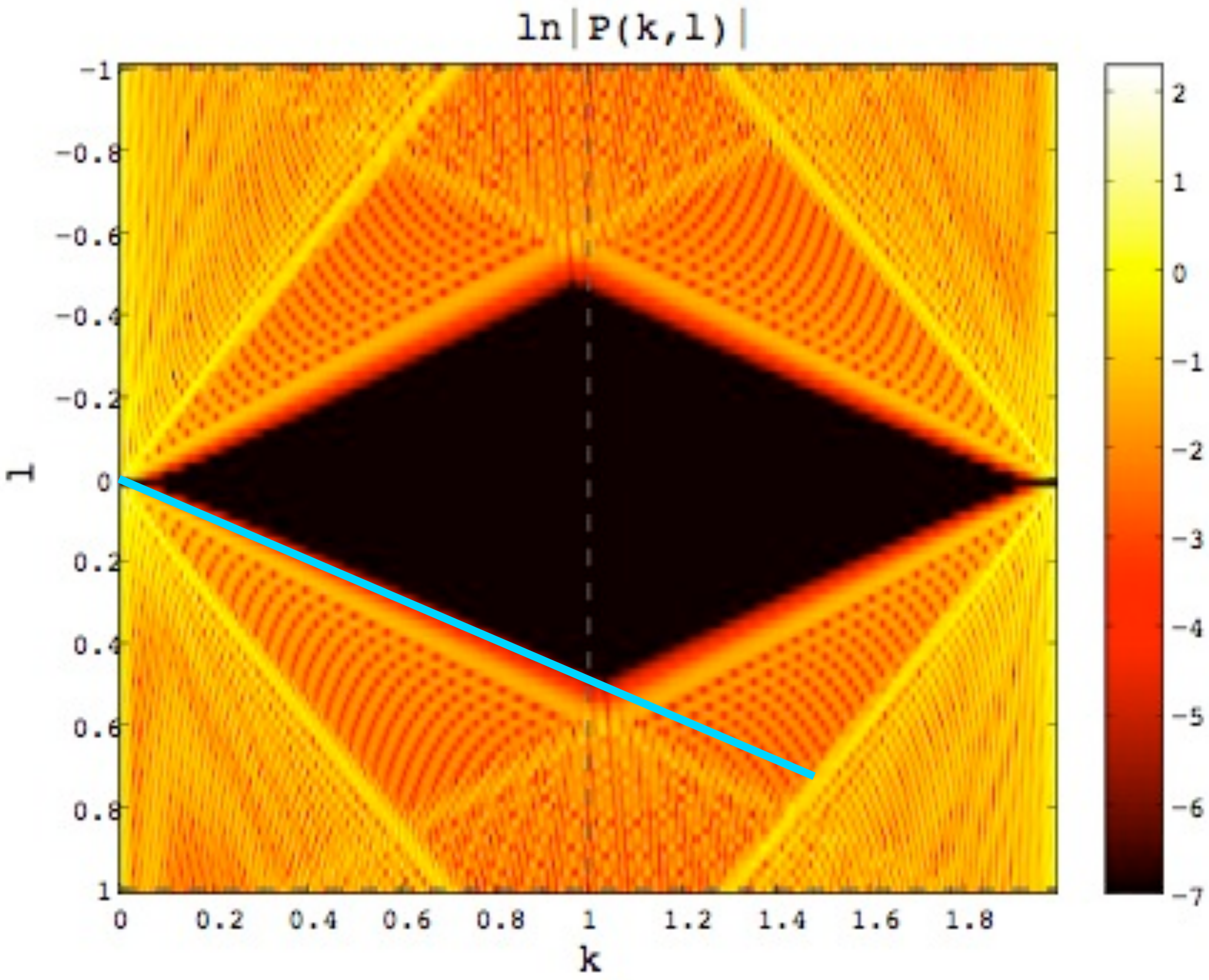

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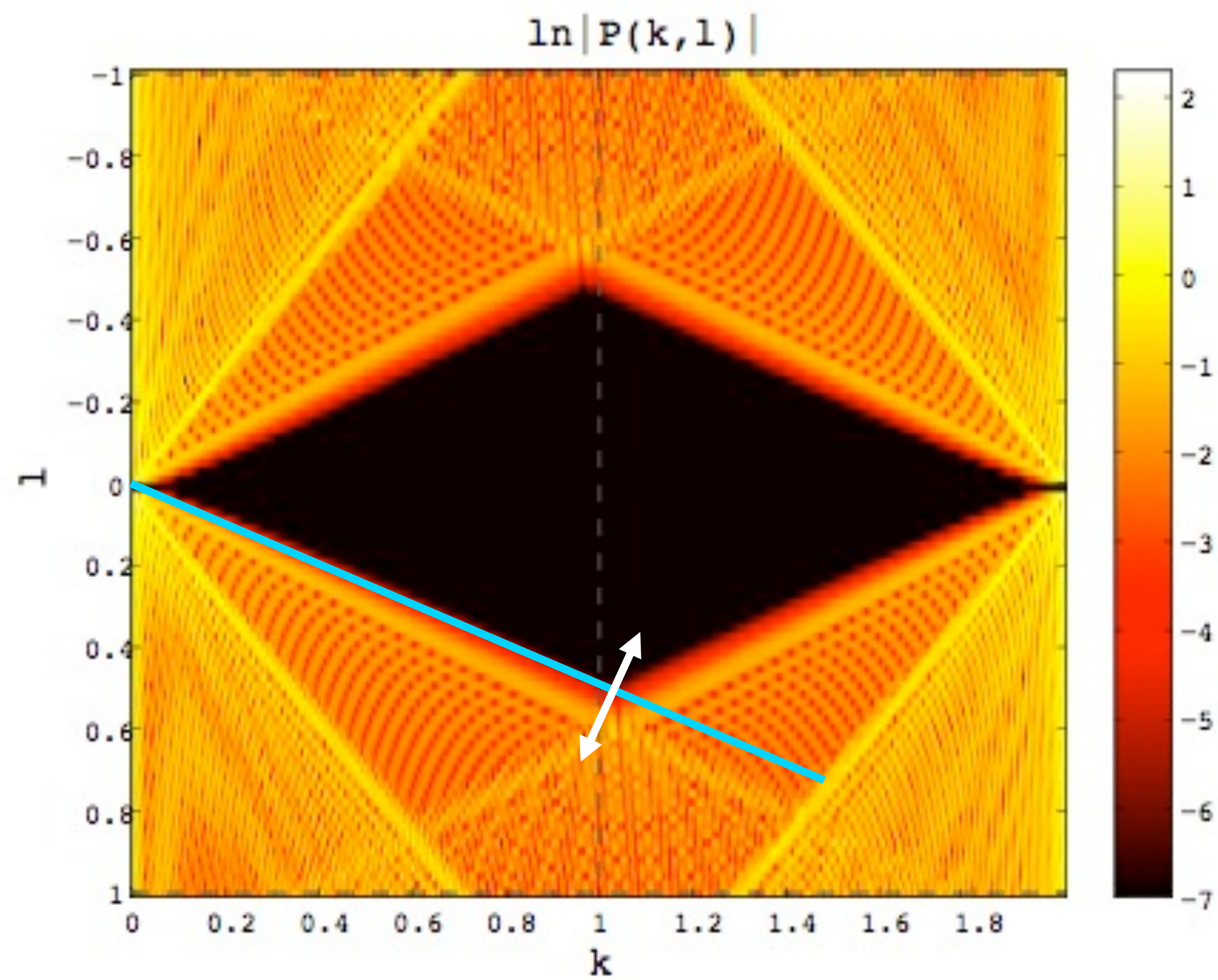

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$$P(k,l) = \int_R e^{i(l\cdot f(x) - k\cdot x)} dx$$

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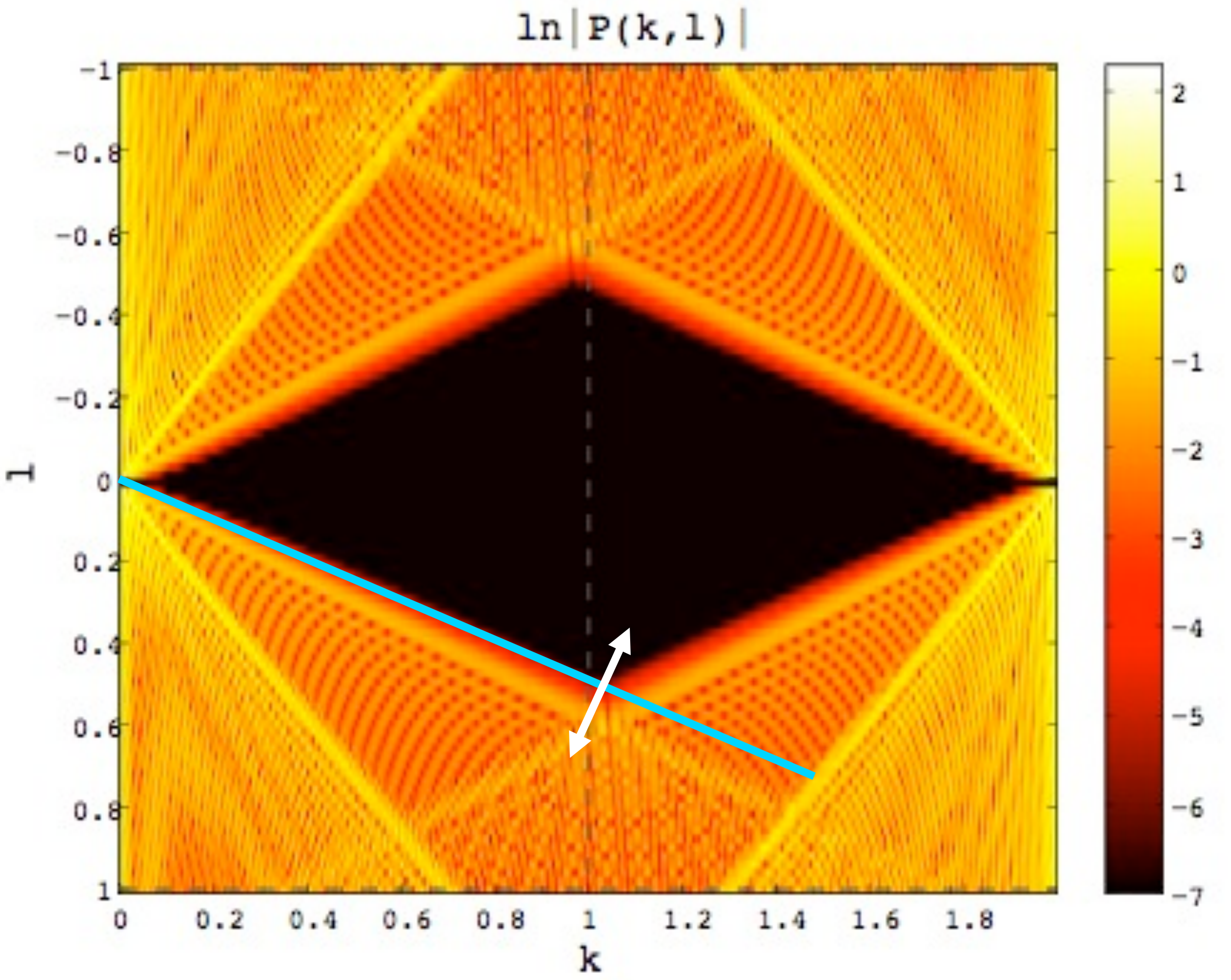
- Exponential drop-off at maximum $l \cdot \max|f'| = k$



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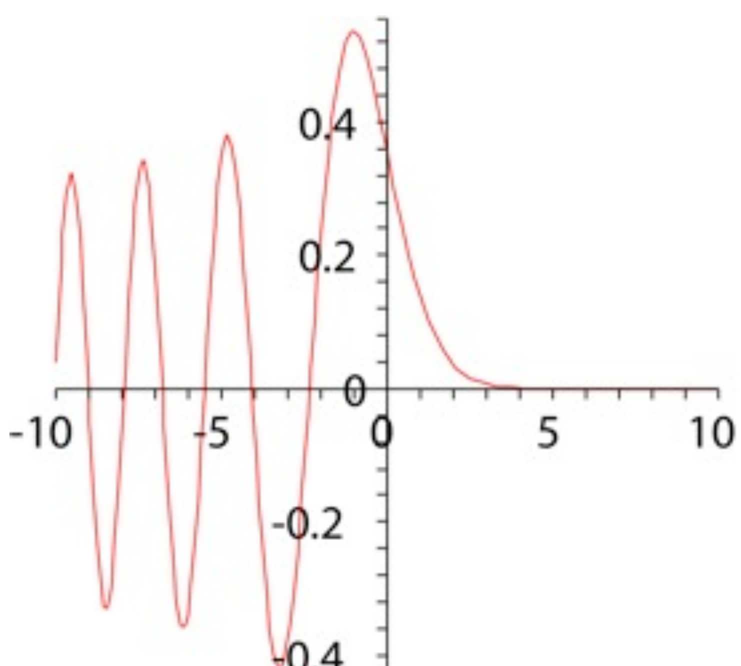
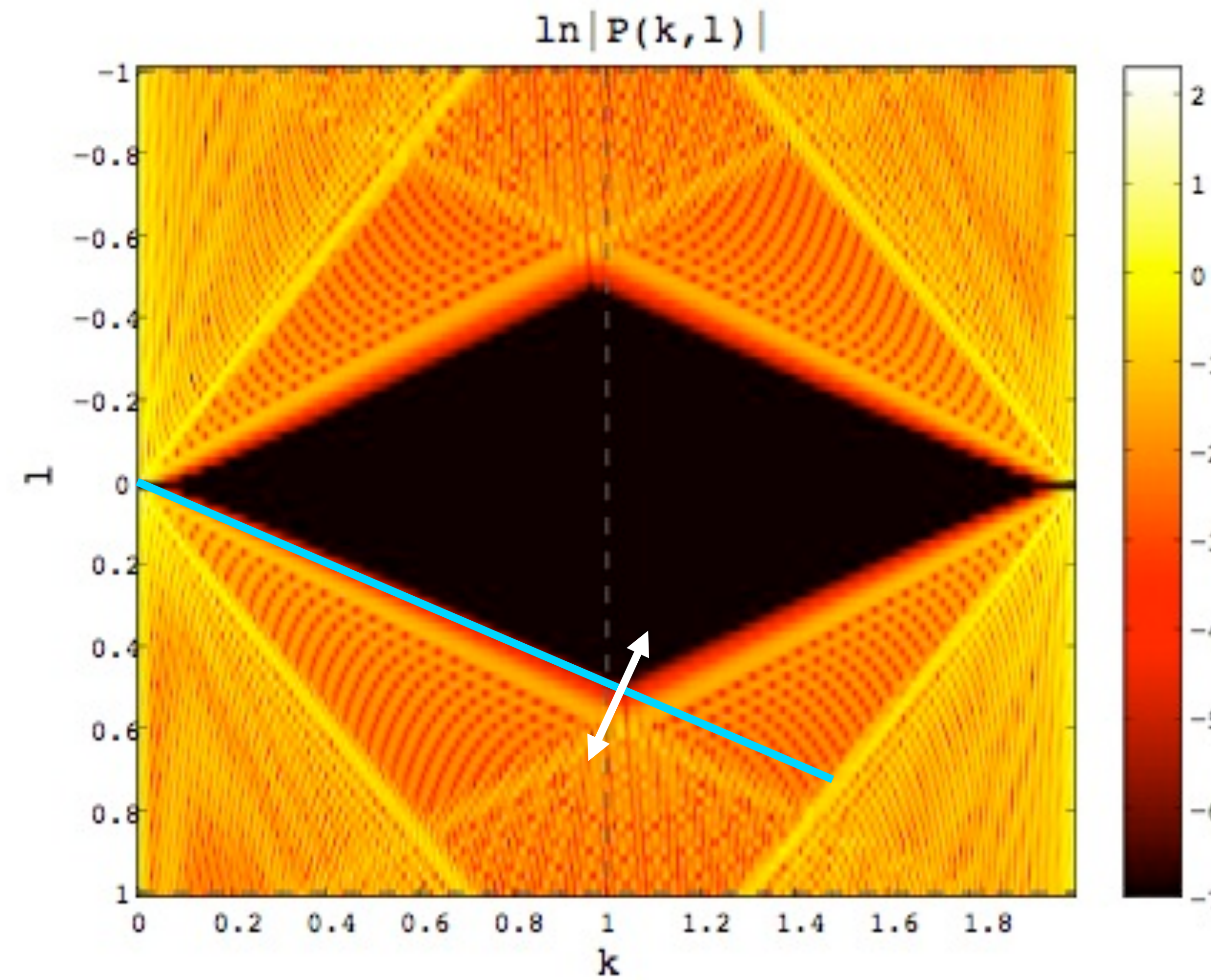
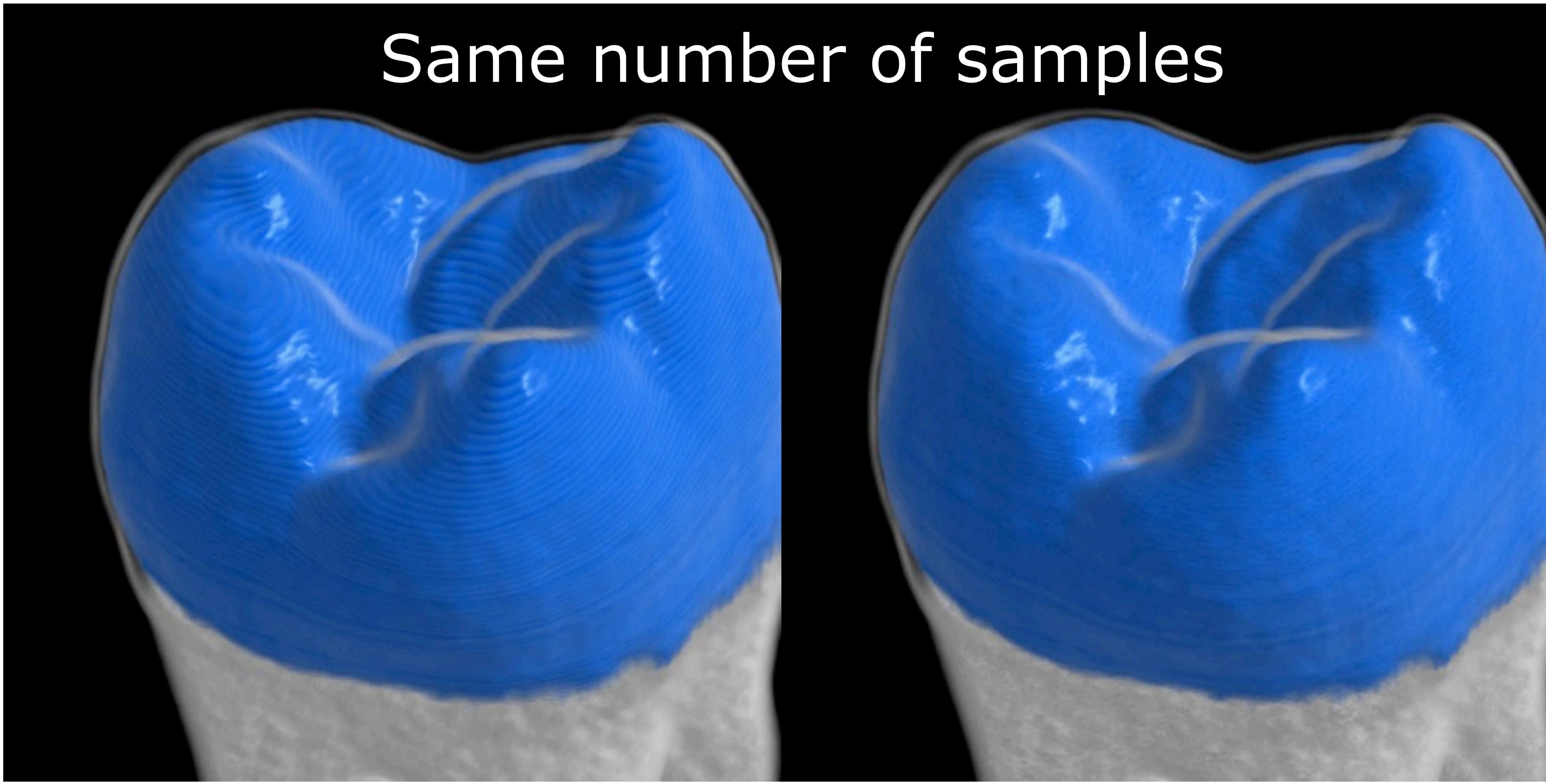


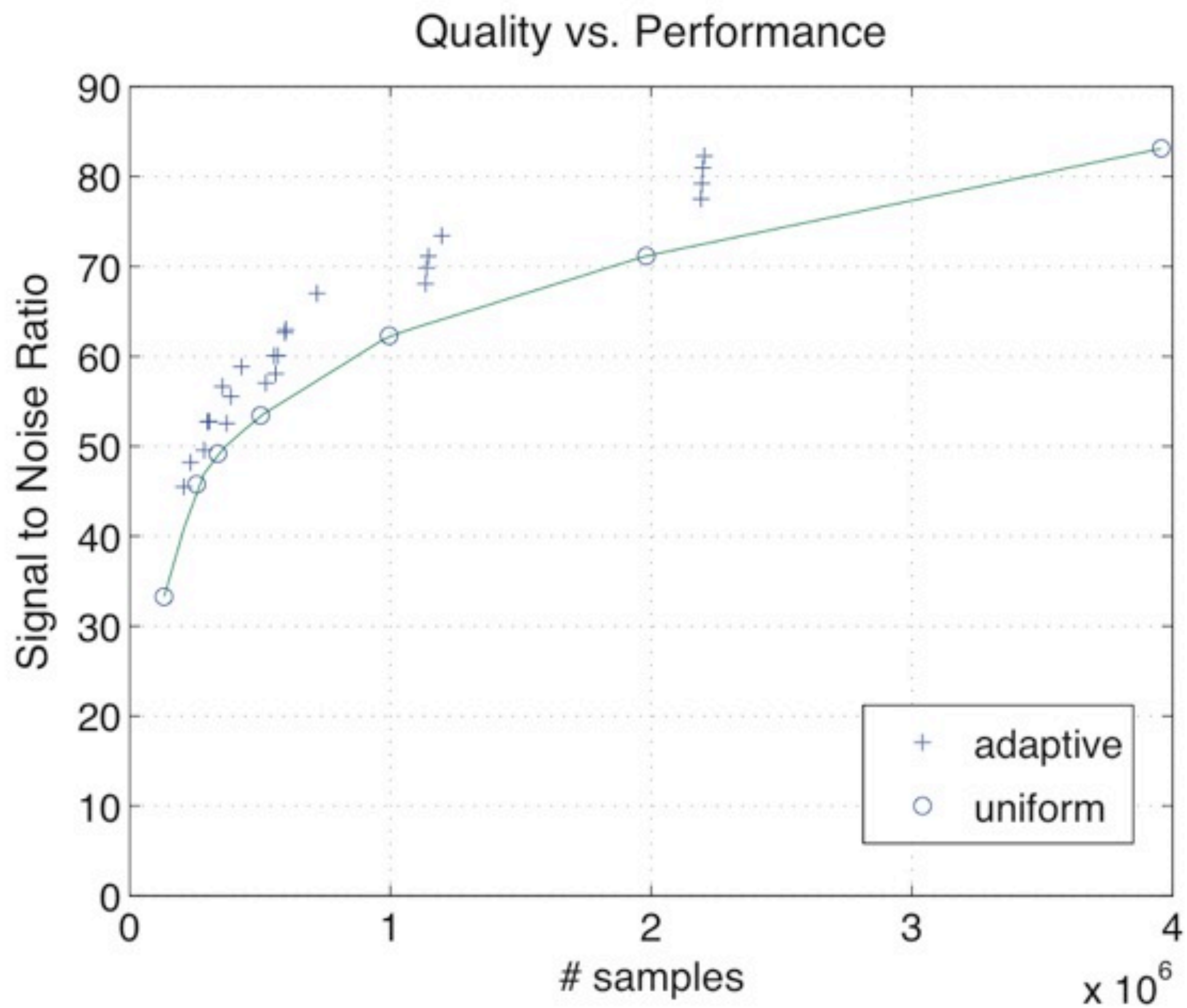


Adaptive Raycasting



Adaptive Raycasting SNR

Ground-truth: computed at a fixed sampling distance of 0.06125



Summary

- Proper sampling of combined signal g(f(x)):

$$v_h = \max(||f'||) \cdot v_g$$

- Solved a fundamental problem of rendering
- Composition is a general data processing operation

Model adjustment at different levels

- User-driven experimentation: Use cases for *paraglide*
- Criteria optimization: Lighting design
- Theoretical analysis: Sampling in volume rendering
- Filling a region: Lattices with rotational dilation
- Summary and conclusion

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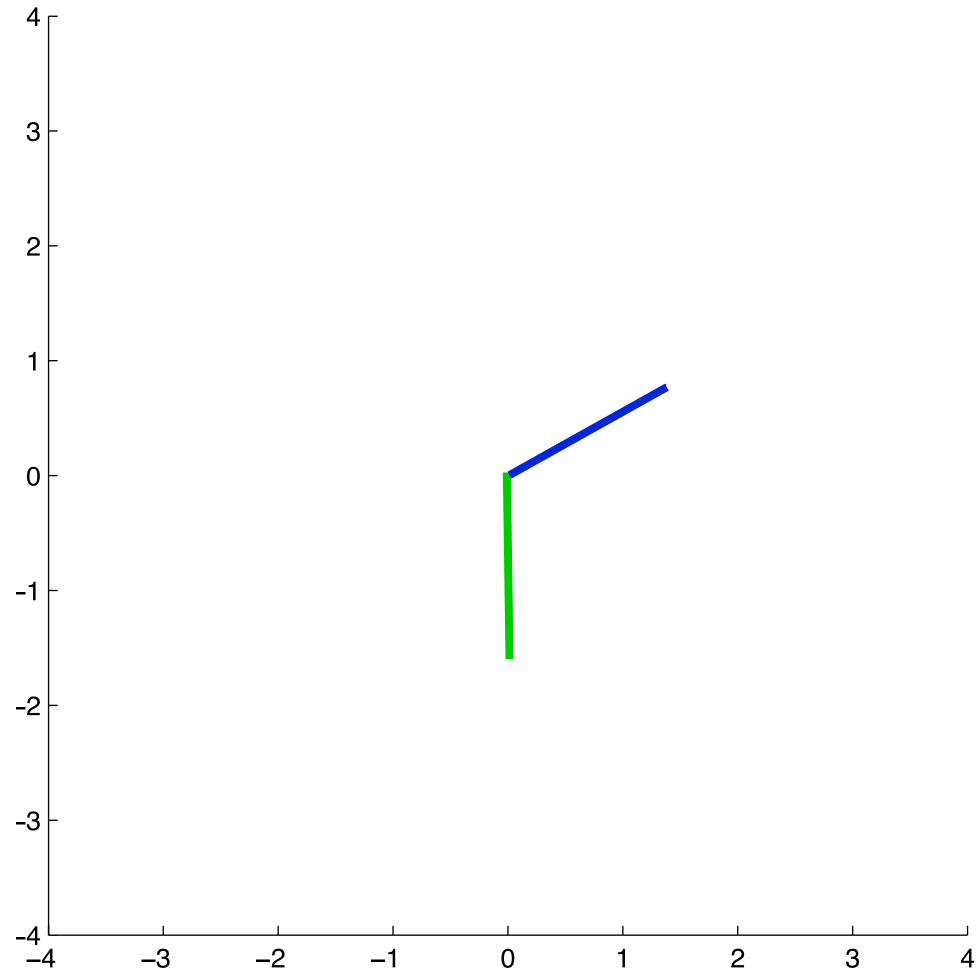
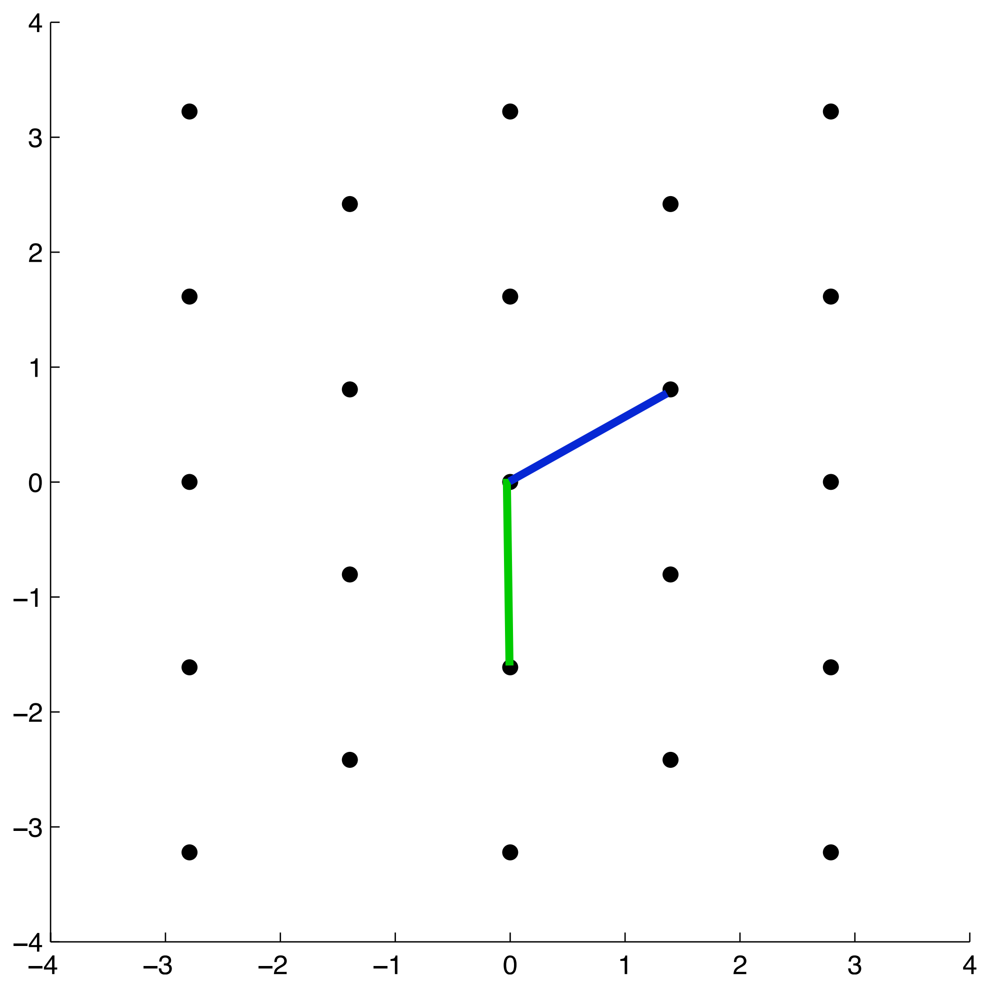
Point lattices

- Definition via basis $\mathbf{R}$

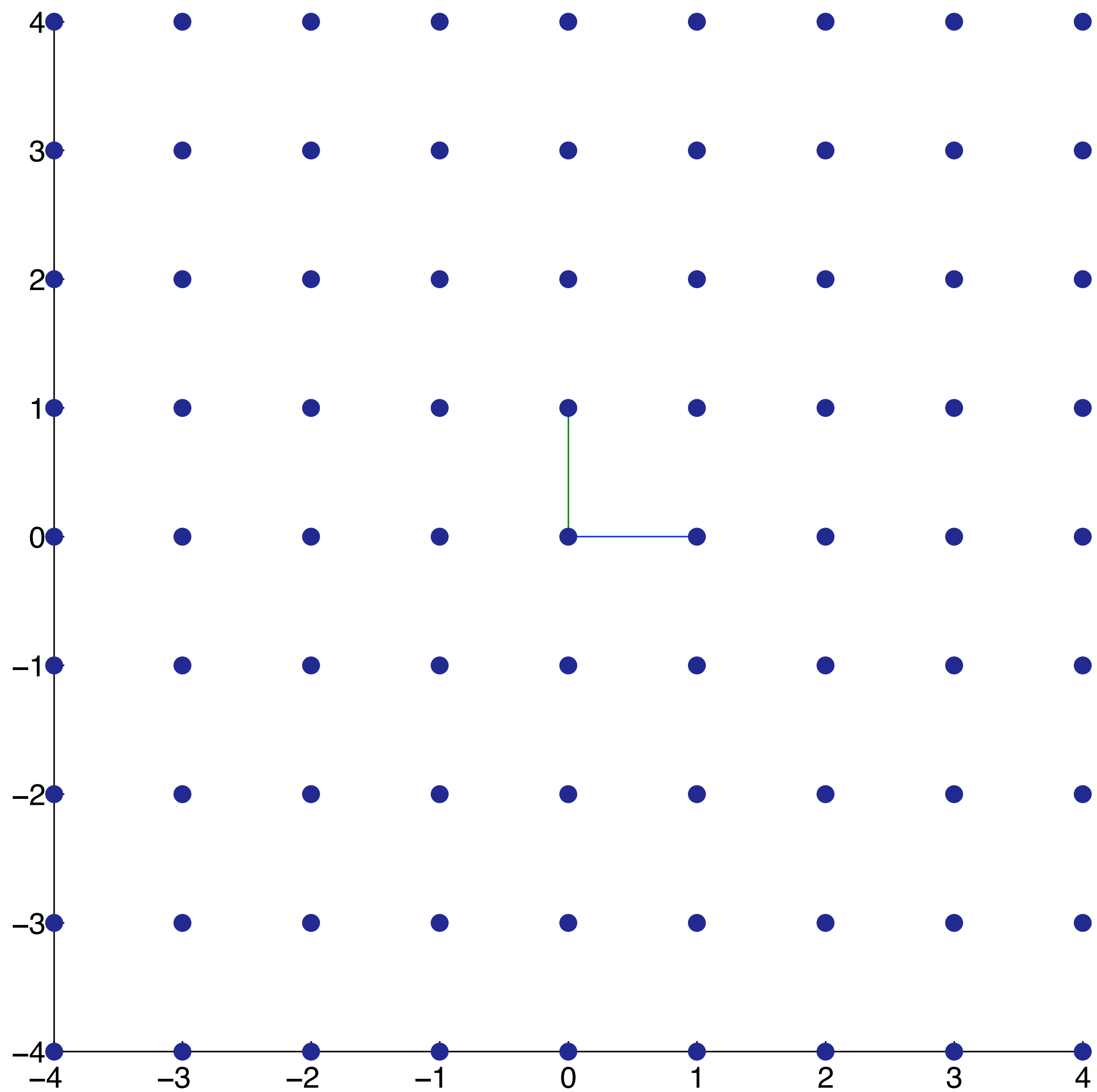


Point lattices

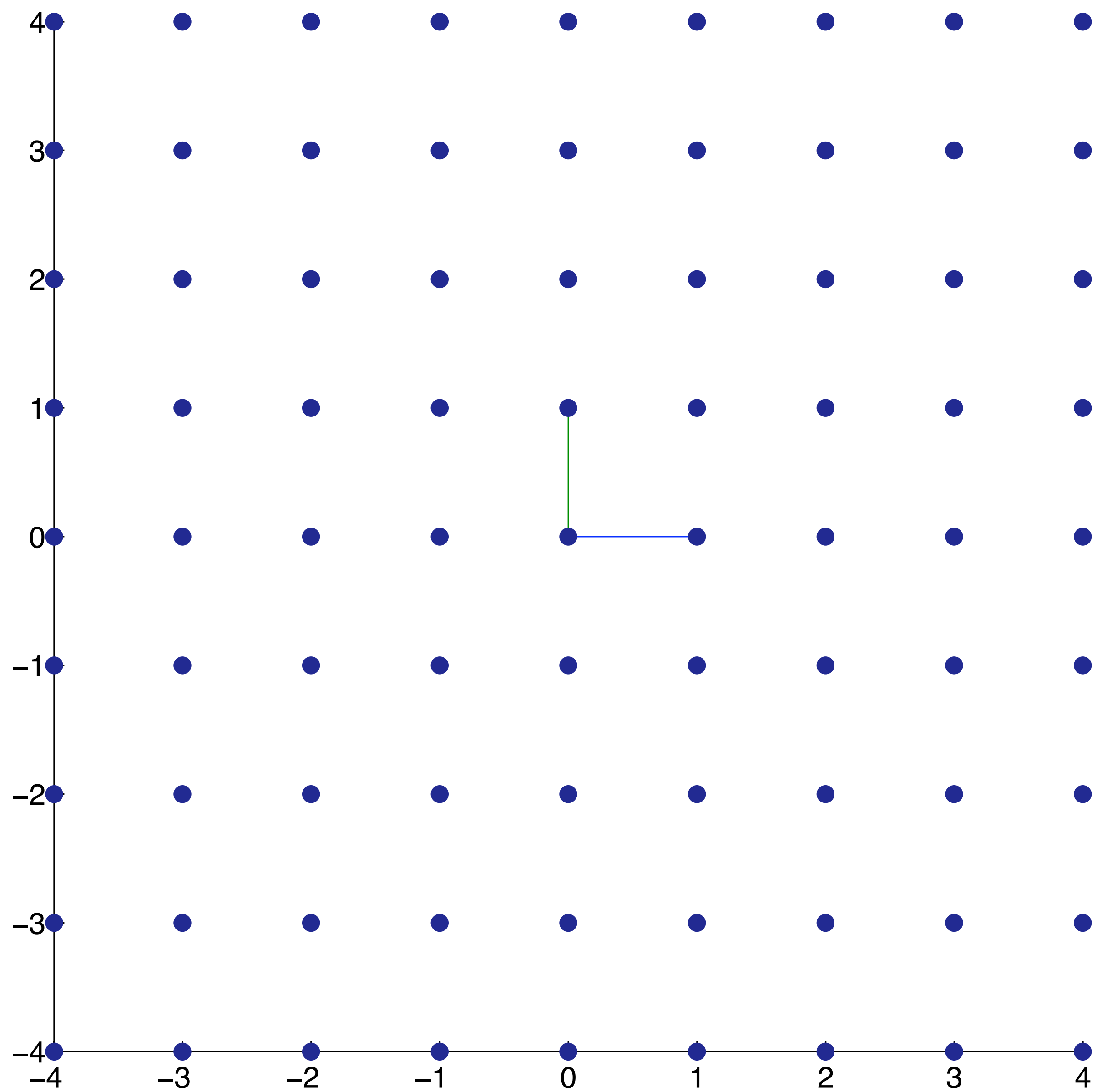
- Definition via basis $\{\mathbf{R}k : k \in \mathbb{Z}^n\}$



$$\mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



$$\mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2\mathbf{I} \qquad \det \mathbf{K} = 2^n = 4$$



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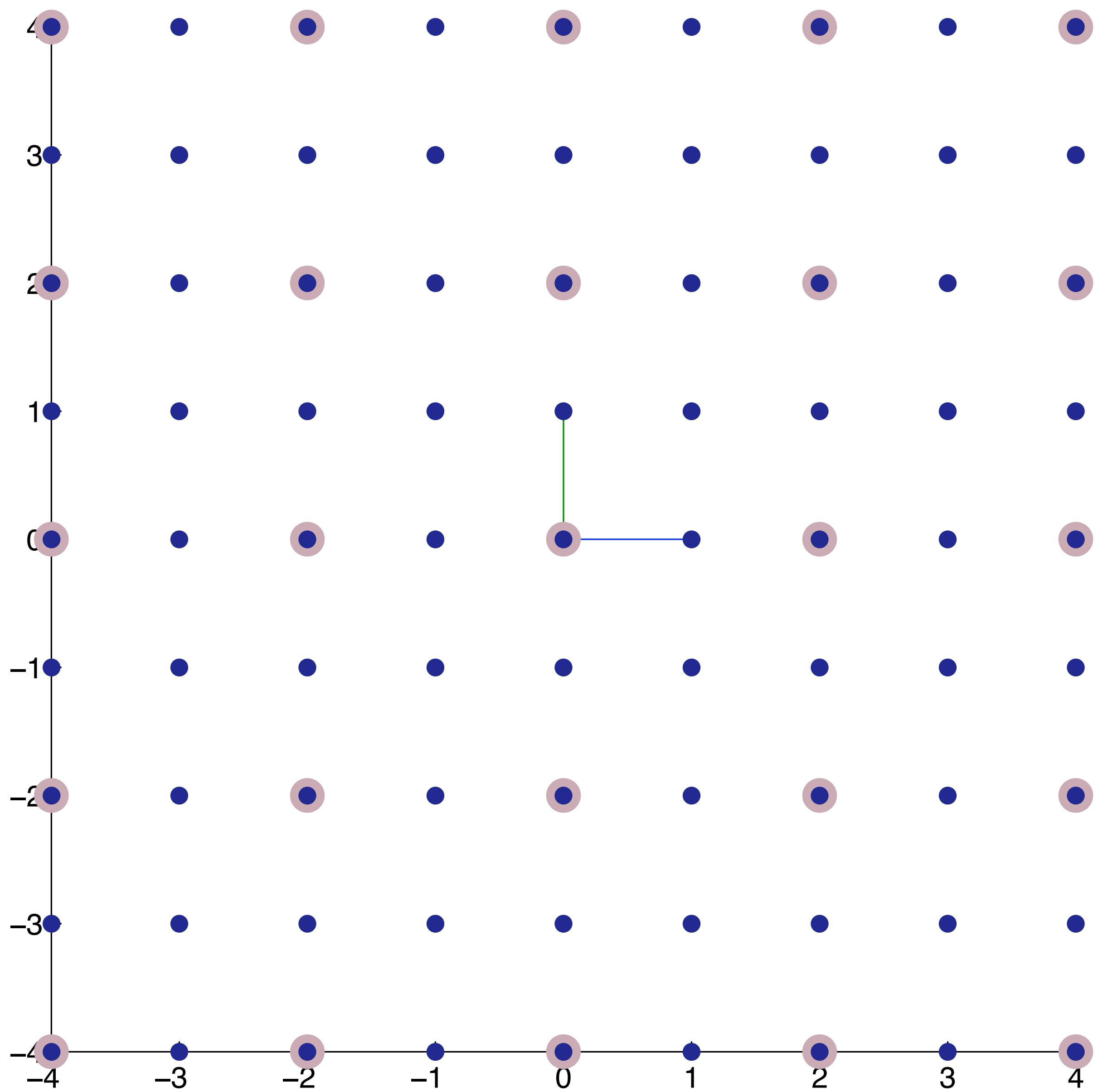

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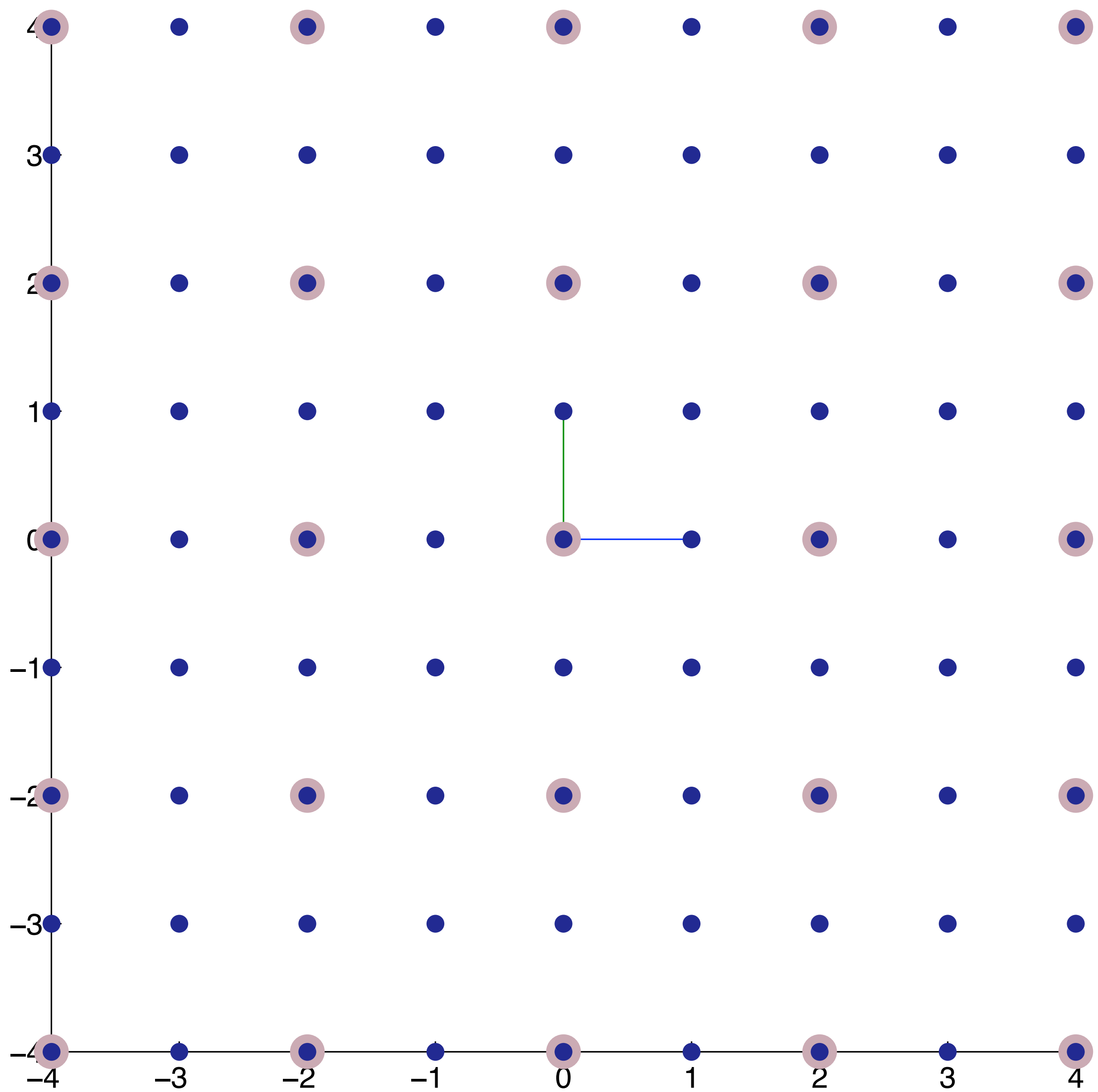

Reduction factor is exponential in n

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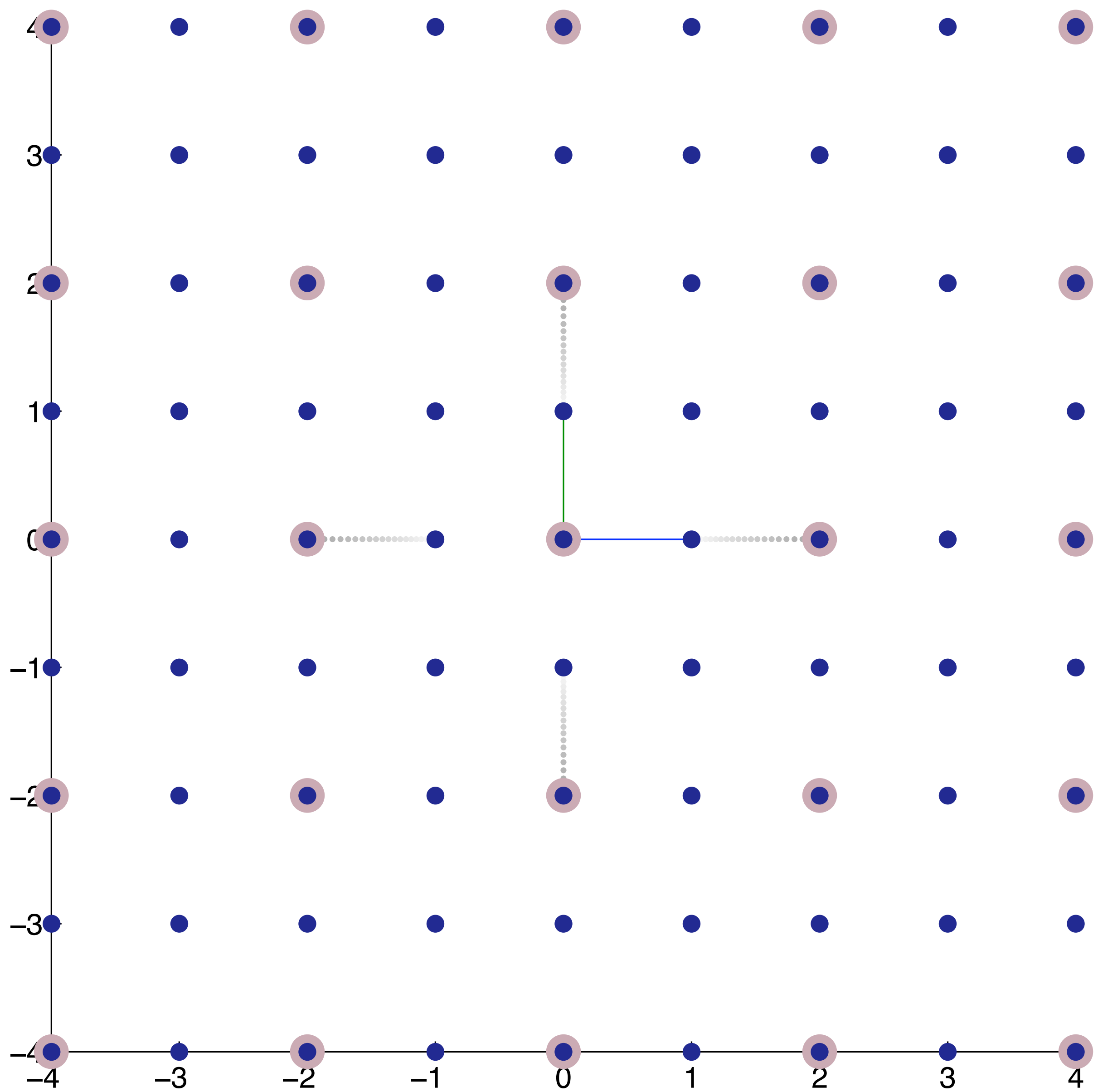

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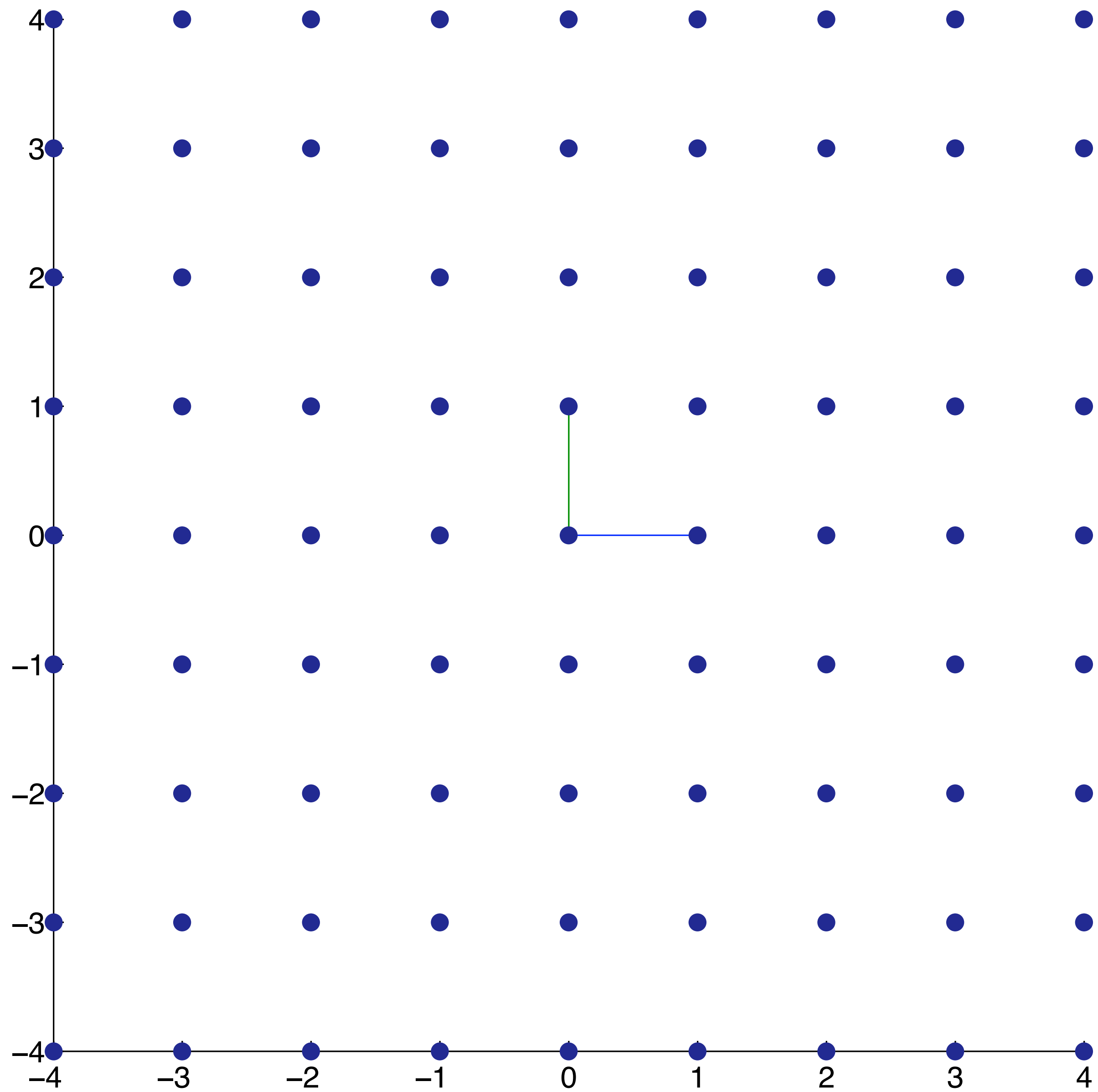

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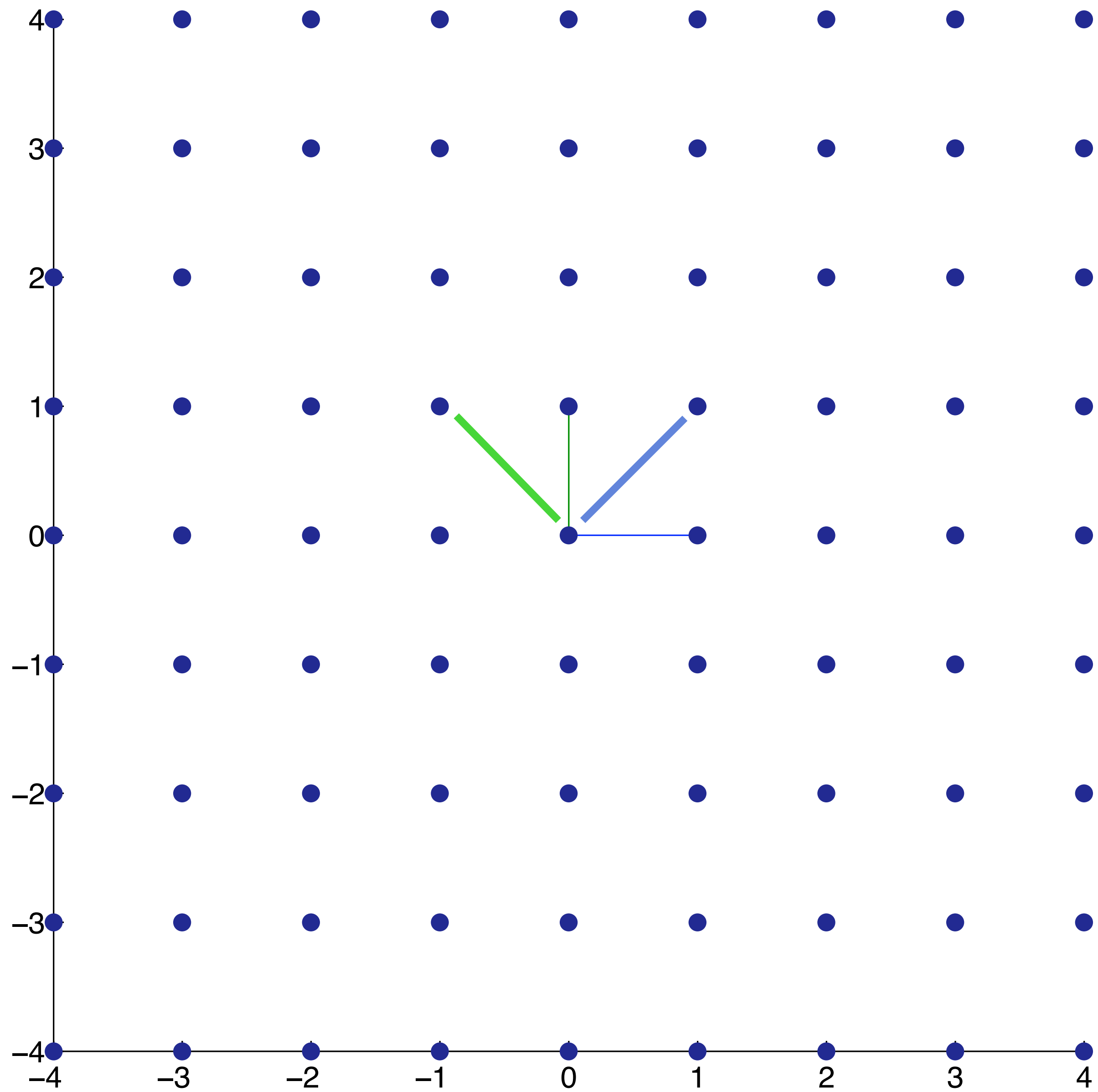

quincunx subsampling

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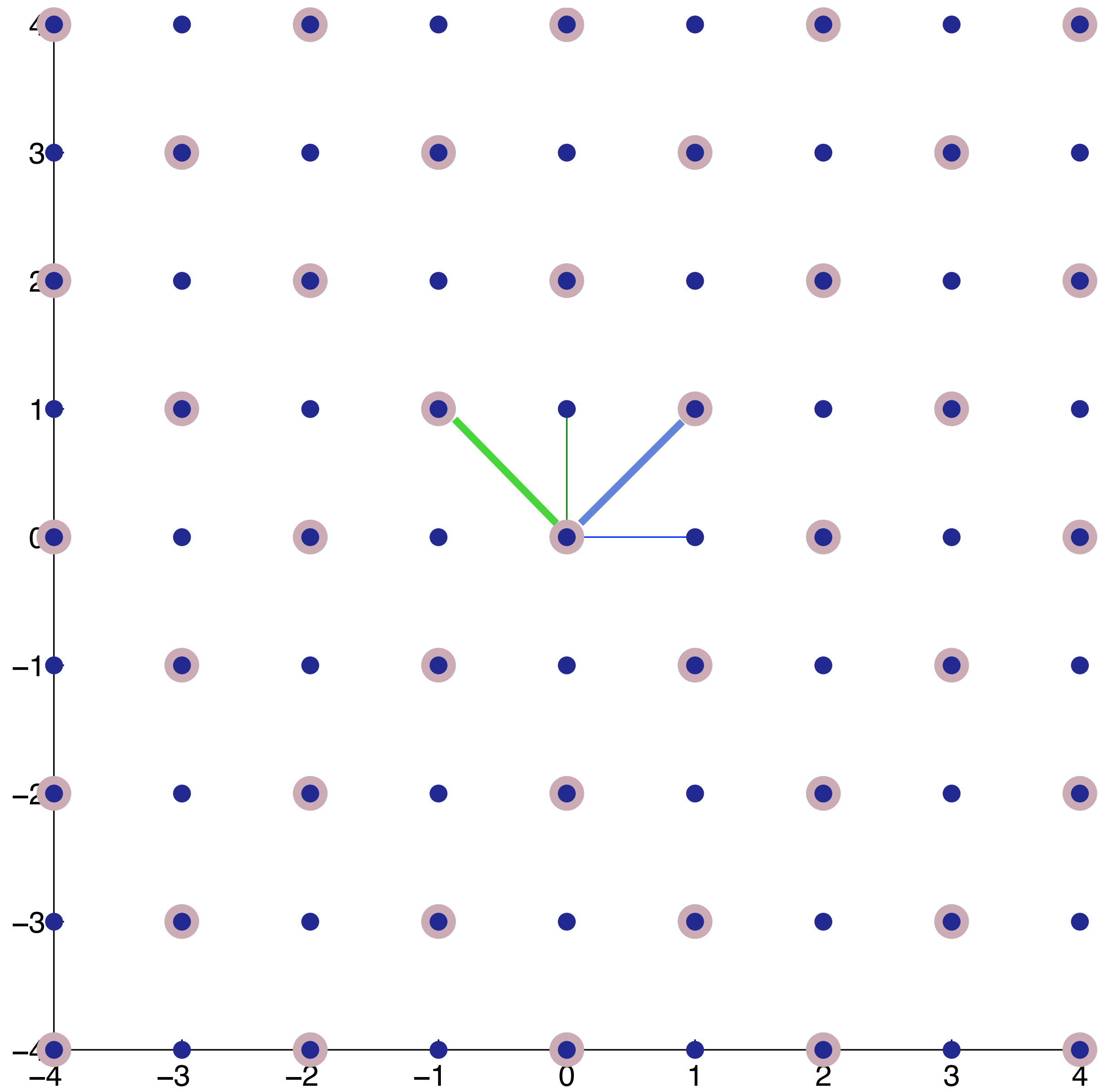

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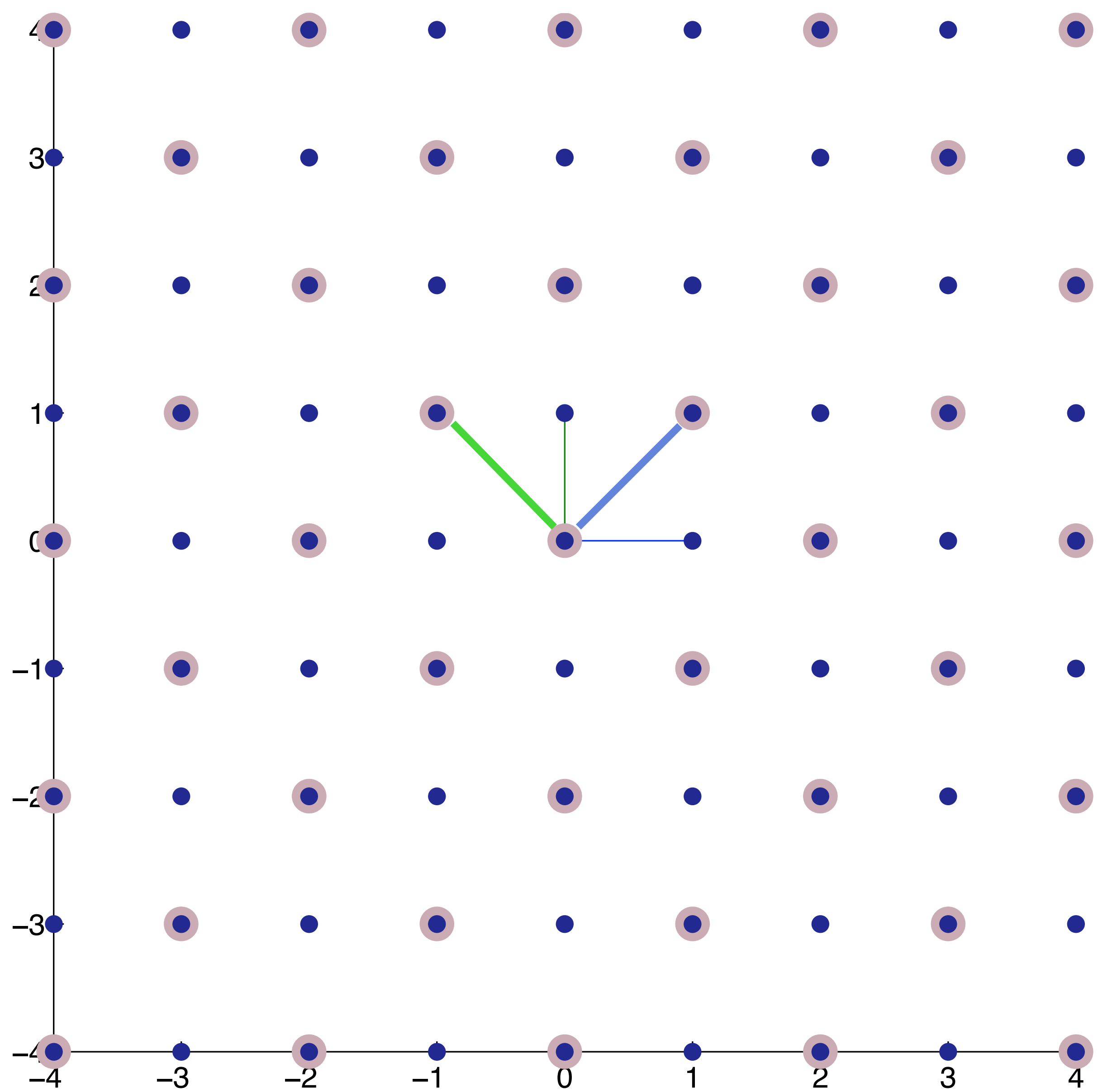

quincunx subsampling

This low rate dilation does not exist for integer lattices with $n > 2$

[Van De Ville, Blu, Unser, SPL 05]

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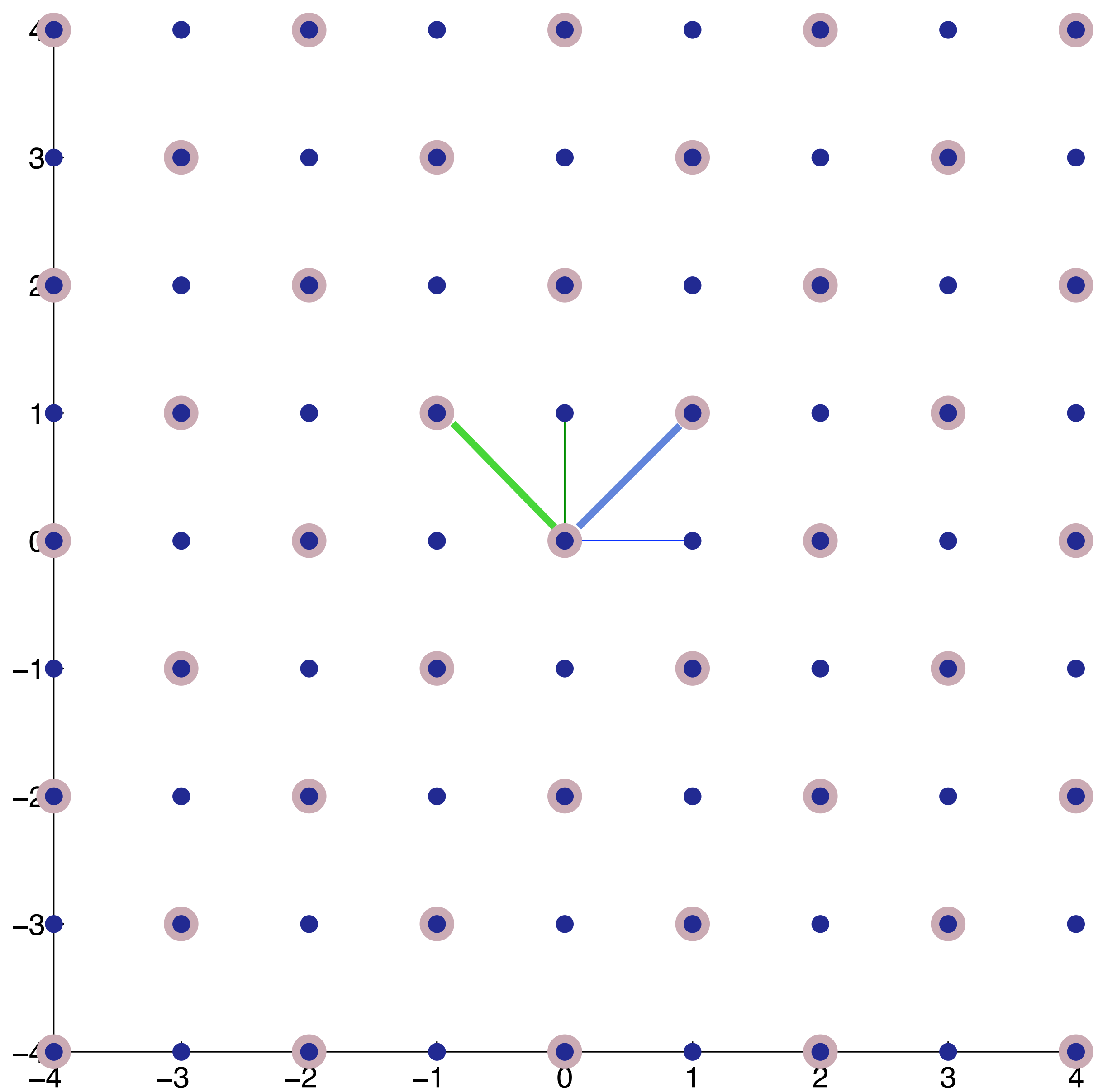

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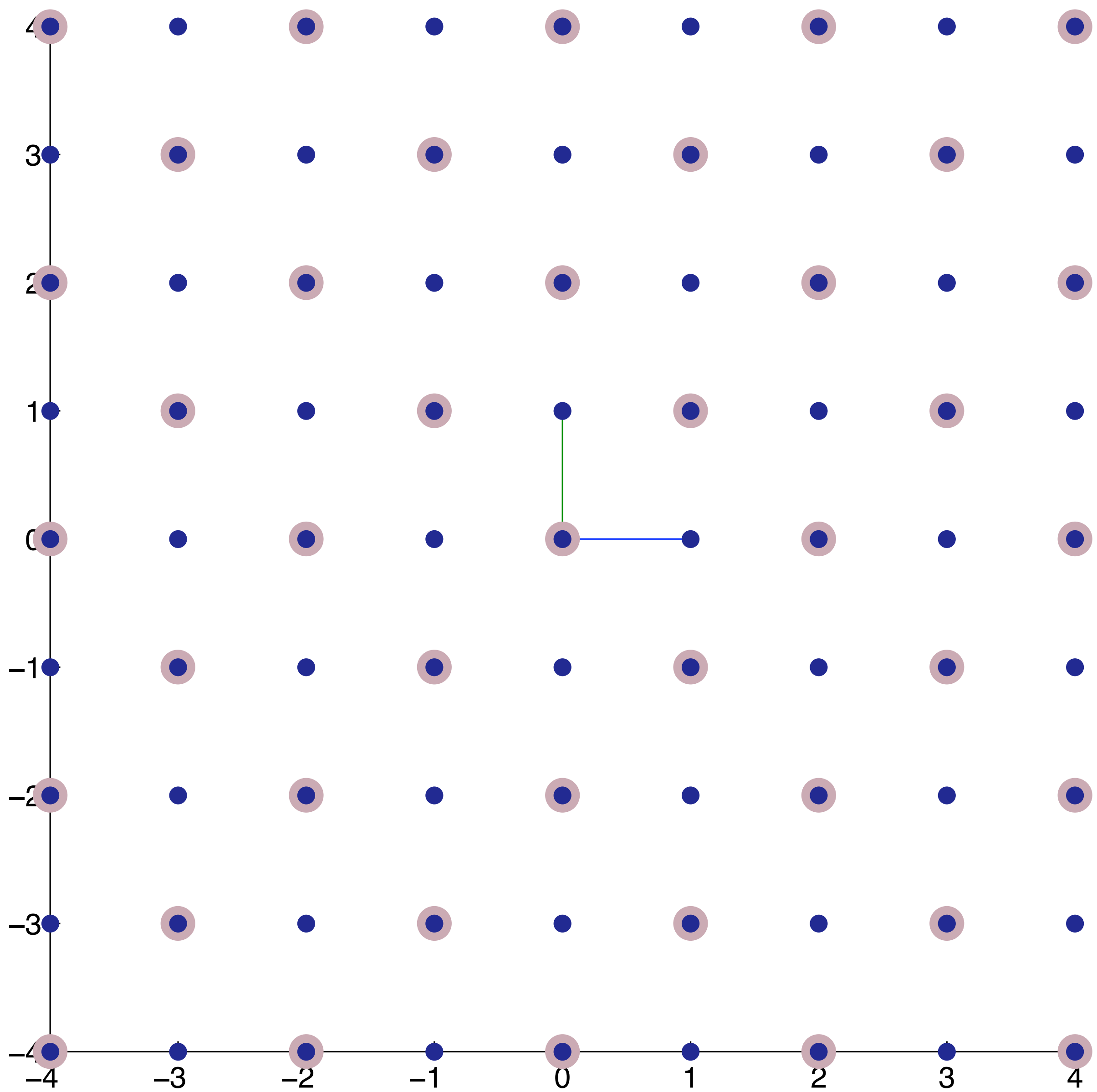
[Van De Ville, Blu, Unser, SPL 05]

However, possible for irrational $\mathbf{R}$!

$$\mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad \det \mathbf{K} = 2$$



quincunx subsampling

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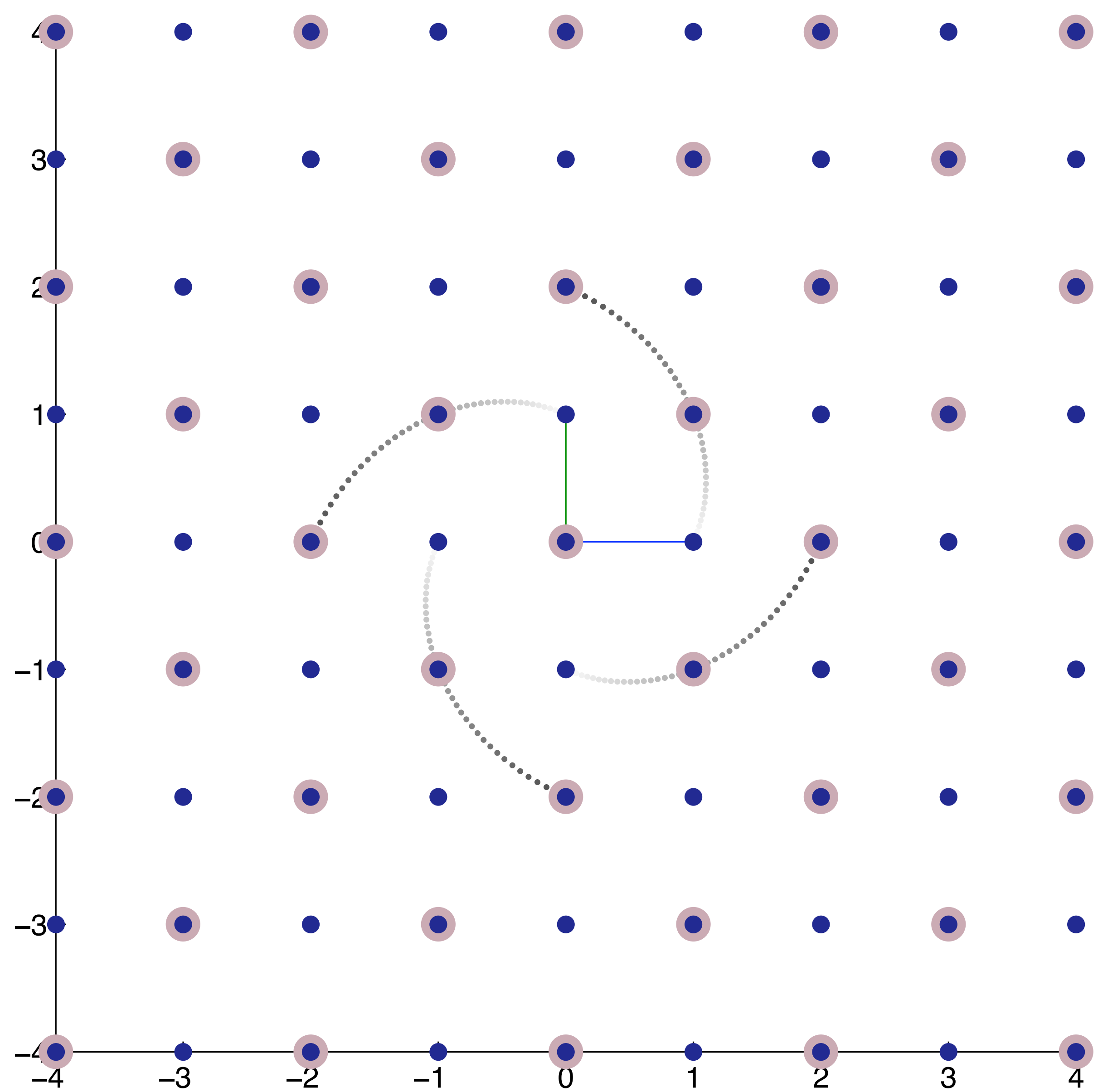

fractional subsampling

$\mathbf{RK}^s$ for $s = 0..2$

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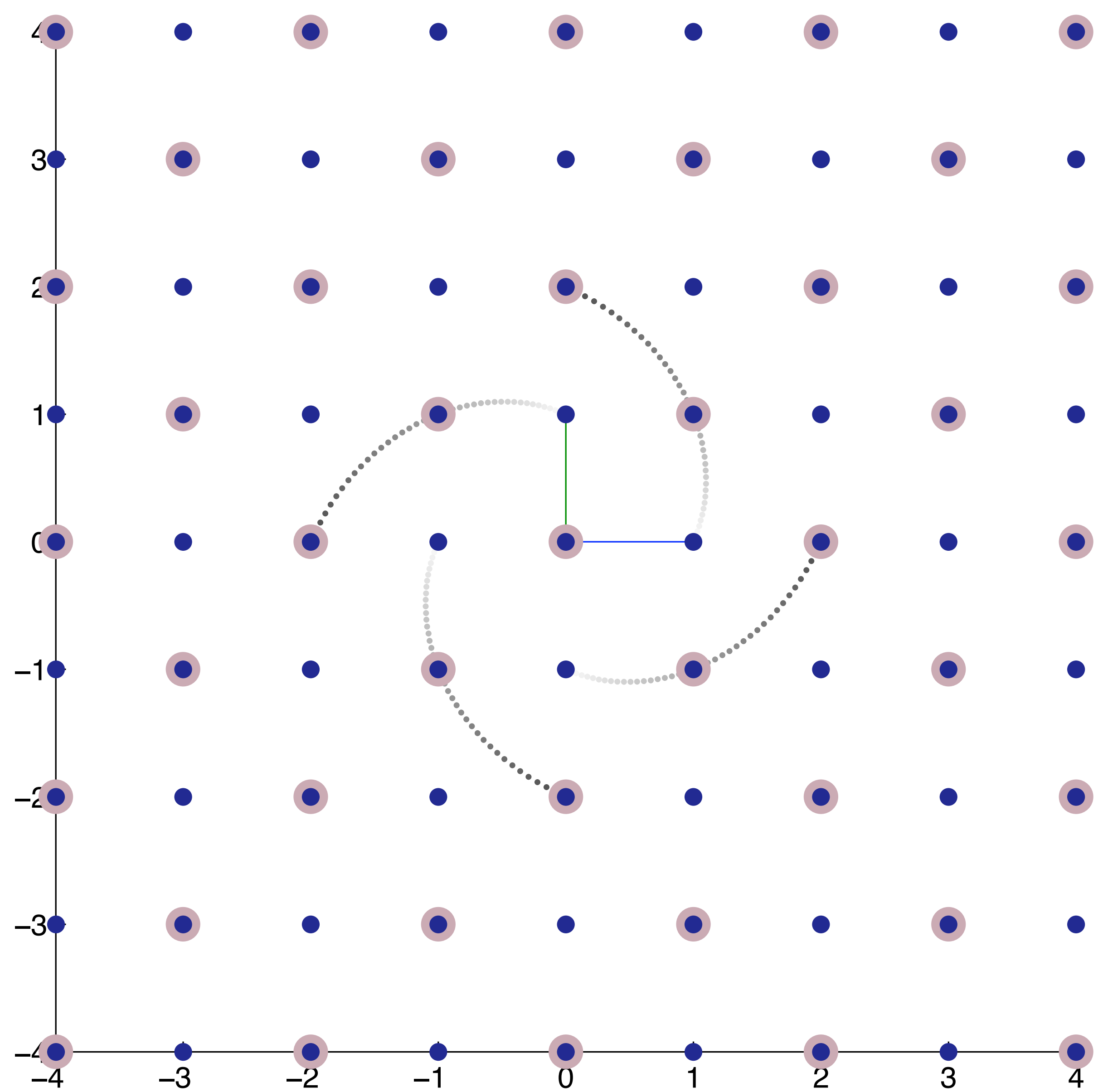

fractional subsampling

$\mathbf{R}\mathbf{K}^s$ for $s = 0..2$

acts like a scaled rotation $\mathbf{QR}$

with $\mathbf{Q}^T\mathbf{Q} = \alpha^2\mathbf{I}$

quincunx subsampling

Construction

Similarity of Q and K

$\mathbf{QR} = \mathbf{RK}$ with $\mathbf{Q}^T\mathbf{Q} = \alpha^2\mathbf{I}$

Similarity of Q and K

$$\mathbf{Q}\mathbf{R} = \mathbf{R}\mathbf{K} \quad \text{with} \quad \mathbf{Q}^T\mathbf{Q} = \alpha^2\mathbf{I}$$

$$\mathbf{R}^{-1}\mathbf{Q}\mathbf{R} = \mathbf{K}$$

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$= \sum_{k=0}^{n} c_k \lambda^k \in \mathbb{Z}[\lambda]$

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and thus agree in eigenvalues and determinant.

Diagonalizing rotation Q

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}\begin{bmatrix} e^{j\theta} & 0 \\ 0 & e^{-j\theta} \end{bmatrix}\begin{bmatrix} 1 & j \\ 1 & -j \end{bmatrix}$$

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Different eigenvalue structure for even and odd dimensionality

$$\mathbf{\Delta} = \begin{bmatrix} e^{j\theta_1} & & & & \\ & e^{-j\theta_1} & & & \\ & & e^{j\theta_2} & & \\ & & & e^{-j\theta_2} & \\ & & & & \ddots \end{bmatrix} \qquad \mathbf{\Delta} = \begin{bmatrix} 1 & & & \\ & e^{j\theta_1} & & \\ & & e^{-j\theta_1} & \\ & & & \ddots \end{bmatrix}$$

With analogue block-wise construction of $\mathbf{J}_n$

Diagonalizing rotation Q

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Different eigenvalue structure for even and odd dimensionality restricts characteristic polynomial:

- n even: $d(\lambda) = \lambda^n + C\lambda^{\frac{n}{2}} + \alpha^n$ with $C^2 < 4\alpha^n$
- n odd: $d(\lambda) = \lambda^n - \alpha^n$

Finding suitable K

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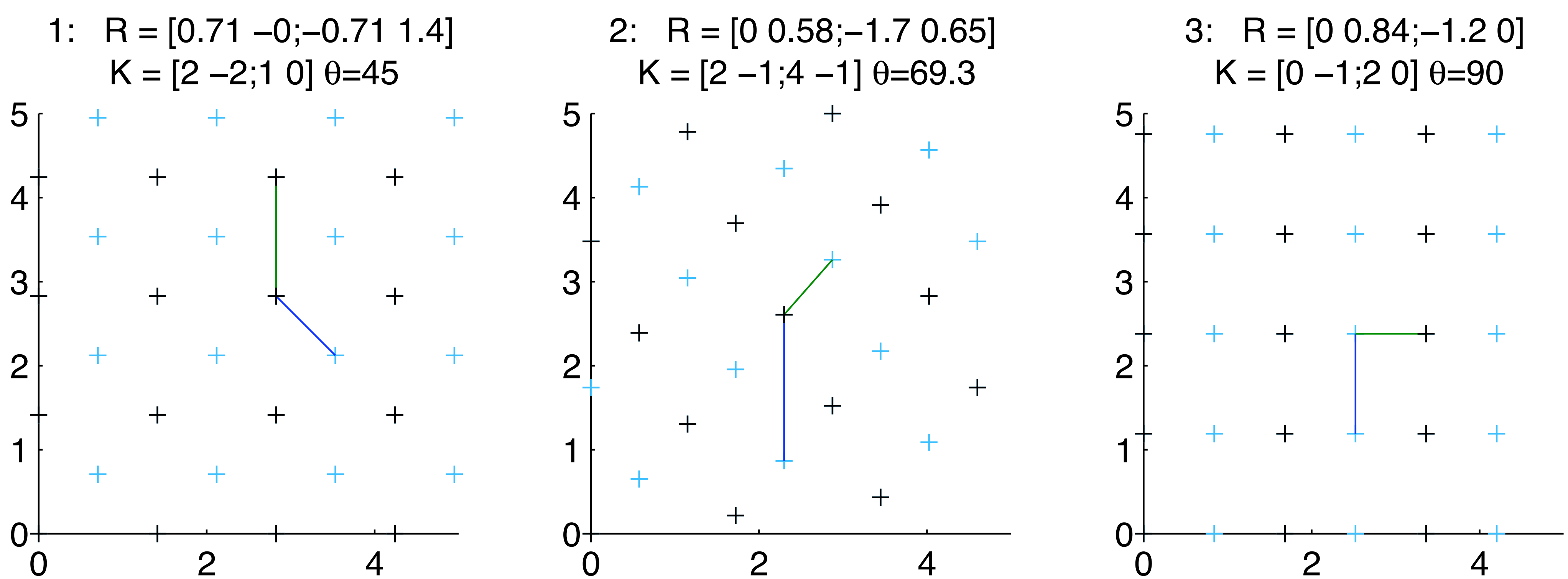
- More with *unimodular* similarity transforms

$$\mathbf{K}_T = \mathbf{T}^{-1}\mathbf{K}\mathbf{T} \text{ with } \det \mathbf{T} = 1 \text{ and } \mathbf{T} \in \mathbb{Z}^{n \times n}$$

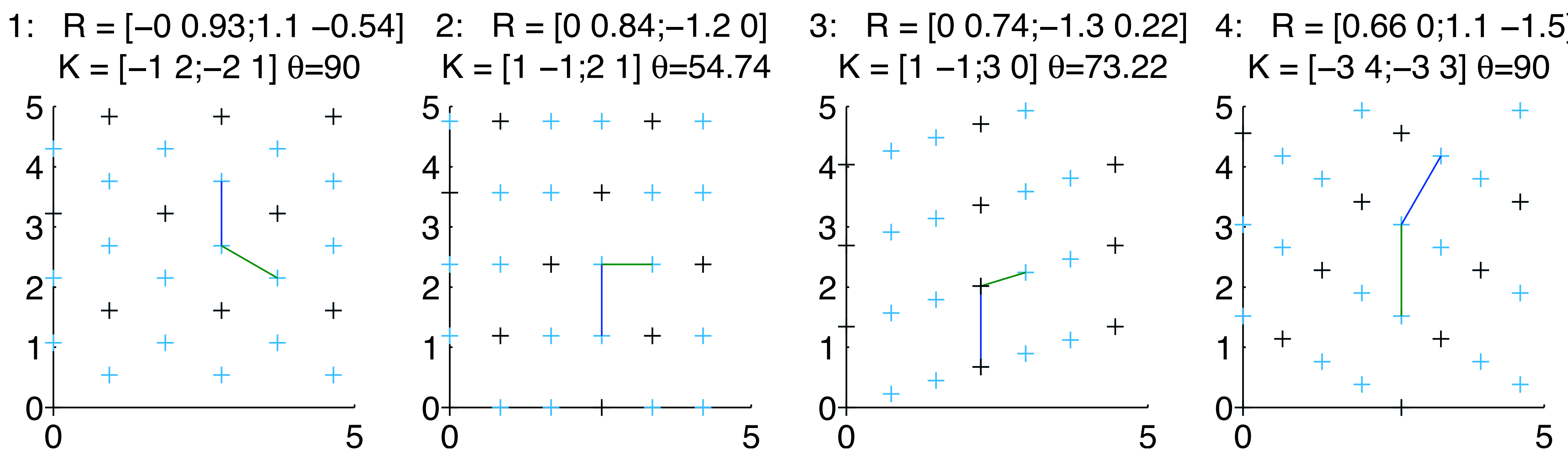
Results

2D



Dilation factor $|\det \mathbf{K}| = 2$

2D



Dilation factor $|\det \mathbf{K}| = 3$

Rotational grid summary

- First time low-rate admissible dilation matrices are available for n>2
- Additional degrees of freedom in the design enable further optimization
- Current results allow optimized constructions up to n=9

Model adjustment at different levels

- User-driven experimentation: Use cases for *paraglide*
- Criteria optimization: Lighting design
- Theoretical analysis: Sampling in volume rendering
- Filling a region: Lattices with rotational dilation
- Summary and conclusion

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Data taxonomy

- *primary:* field measurements
- *secondary:* synthetic data or human input
- *tertiary:* rules provided by theoretical study or statistical inference

Model adjustment at different levels

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Model adjustment at different levels

User input

- User-driven experimentation: Use cases for *paraglide*
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Theoretical input

Acknowledgements

- Torsten Möller and Derek Bingham
- GrUVi-Lab members
- Collaborators: SFU -CS, -Maths, and -Stats
  BIG @ EPFL
- Funding: SFU, Precarn, and NSERC

Thank you! Questions?